

LIMIT DISTRIBUTIONS OF RANDOM MATRICES

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ABSTRACT. We study limit joint distributions of the ensemble of symmetric blocks of independent Gaussian random matrices with block-identical variances (Gaussian Symmetric Block Ensemble). Our approach is based on the concept of noncommutative independence called matricial freeness which reminds freeness in free probability, but describes the asymptotics of symmetric blocks rather than that of the whole matrices. The main result can be viewed as a block refinement of Voiculescu's asymptotic freeness of the ensemble of independent Gaussian random matrices. We show that this concept provides a unified framework for studying the asymptotic distributions of sums and products of independent Gaussian random matrices, including random matrices of Wishart type and random matrix models for free Bessel laws. Moreover, it leads to random matrix models for boolean independence, monotone independence and s-freeness.

1. INTRODUCTION AND MAIN RESULTS

One of the most important features of free probability is its close relation to random matrices. It has been shown by Voiculescu that independent Hermitian Gaussian random matrices are asymptotically free [32]. This result shows that the concept of freeness, or free independence, is not only a noncommutative analog of classical independence, but is also fundamental in the random matrix theory, which puts the classical result of Wigner on the semicircle law as the limit distribution of certain symmetric random matrices [36] in an entirely new perspective. Moreover, it is universal in view of Dykema's generalization to non-Gaussian random matrices [12].

In particular, if we are given an ensemble of independent Hermitian $n \times n$ random matrices

$$\{Y(u, n) : u \in \mathcal{U}\}$$

whose entries are suitably normalized and independent complex Gaussian random variables for each natural n , then

$$\lim_{n \rightarrow \infty} \tau(n)(Y(u_1, n) \dots Y(u_m, n)) = \Phi(\omega(u_1) \dots \omega(u_m))$$

for any $u_1, \dots, u_m \in \mathcal{U}$, where $\{\omega(u) : u \in \mathcal{U}\}$ is a semicircular family of *free Gaussian operators* living in the free Fock space with the vacuum state Φ and $\tau(n)$ is the normalized trace composed with classical expectation. This realization of the limit distribution gives a fundamental relation between random matrices and operator algebras.

Consequently, if \mathcal{U} is finite, the moments of the sum

$$Y(n) = \sum_{u \in \mathcal{U}} Y(u, n)$$

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under $\tau(n)$ converge to the moments of the sum of free Gaussian operators

$$\omega = \sum_{u \in \mathcal{U}} \omega(u)$$

and can be described in terms of the free additive convolution of semicircle laws

$$\sigma = \sigma(1) \boxplus \sigma(2) \boxplus \dots \boxplus \sigma(m)$$

where m is the cardinality of \mathcal{U} . The free additive convolution, introduced by Voiculescu [33] for compactly supported measures on the real line and generalized by Bercovici and Voiculescu [7] to measures with non-compact support, turned out to be an effective tool in the study of asymptotic spectral distributions of sums of independent Hermitian random matrices.

The basic original random matrix model studied by Voiculescu corresponds to independent complex Gaussian variables, where the entries $Y_{i,j}(u, n)$ of each matrix $Y(u, n)$ satisfy the Hermiticity condition $Y_{i,j}(u, n) = \overline{Y_{j,i}(u, n)}$ as well as

$$\mathbb{E}(Y_{i,j}(u, n)) = 0 \quad \text{and} \quad \mathbb{E}(|Y_{i,j}(u, n)|^2) = 1/n$$

for any $1 \leq i, j \leq n$ and $u \in \mathcal{U}$. If we relax the second assumption, the scalar-valued free probability is no longer sufficient to describe the asymptotics of Gaussian random matrices. One approach is to use the operator-valued free probability, as in the work of Shlakhtyenko [30], who studied the asymptotics of Gaussian random band matrices and proved that they are asymptotically free with amalgamation over some commutative algebra.

Our approach is based on the decomposition of independent Hermitian Gaussian random matrices $Y(u, n)$ with block-identical variances of $|Y_{i,j}(u, n)|$ into symmetric blocks $T_{p,q}(u, n)$, where $u \in \mathcal{U}$ and $n \in \mathbb{N}$, namely

$$Y(u, n) = \sum_{1 \leq p \leq q \leq r} T_{p,q}(u, n)$$

where each $T_{p,q}(u, n)$ is built from the entries $Y_{i,j}(u, n)$ indexed by pairs (i, j) taken from the union $(N_p \times N_q) \cup (N_q \times N_p)$, where

$$[n] = N_1 \cup \dots \cup N_r, \quad \text{with} \quad \lim_{n \rightarrow \infty} \frac{|N_q|}{n} = d_q \geq 0 \quad \text{for any } q,$$

is the partition of the set $[n] := \{1, 2, \dots, n\}$ into disjoint non-empty subsets (the fact that they depend on n is suppressed in the notation).

At the same time, we decompose each semicircular Gaussian operator in terms of the corresponding $r \times r$ array of *matricially free Gaussian operators* [23], namely

$$\omega(u) = \sum_{p,q=1}^r \omega_{p,q}(u),$$

living in the matricially free Fock space of tracial type, similar to the matricially free Fock space introduced in [23], in which we distinguish a family of states $\{\Psi_1, \dots, \Psi_r\}$, used to build the array $(\Psi_{p,q})$ by setting $\Psi_{p,q} = \Psi_q$.

Nevertheless, in order to reproduce the limit distributions of symmetric blocks in the case when the variances of $|Y_{i,j}(u, n)|$ are only block-identical rather than identical, we

still need to rescale the matricially free Gaussian operators in the above expressions. The corresponding arrays of distributions

$$[\sigma(u)] = \begin{pmatrix} \sigma_{1,1}(u) & \kappa_{1,2}(u) & \dots & \kappa_{1,r}(u) \\ \kappa_{2,1}(u) & \sigma_{2,2}(u) & \dots & \kappa_{2,r}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{r,1}(u) & \kappa_{r,2}(u) & \dots & \sigma_{r,r}(u) \end{pmatrix},$$

consisting of semicircle laws $\sigma_{q,q}(u)$ and Bernoulli laws $\kappa_{p,q}(u)$, replace the semicircle laws $\sigma(u)$ and play the role of *matricial semicircle laws*. In turn, the rescaled matricially free Gaussians ($\omega_{p,q}(u)$) replace free Gaussians $\omega(u)$ and thus the corresponding sums become more general objects called *random pseudomatrices*.

In any case, it is the ensemble of *symmetrized Gaussian operators* defined by

$$\hat{\omega}_{p,q}(u) = \begin{cases} \omega_{q,q}(u) & \text{if } p = q \\ \omega_{p,q}(u) + \omega_{q,p}(u) & \text{if } p \neq q \end{cases}$$

which gives the operatorial realizations of the limit joint distributions of the ensemble of symmetric blocks of the family $\{Y(u, n) : u \in \mathcal{U}\}$. The array of distributions corresponding to each $(\hat{\omega}_{p,q}(u))$ consists of diagonal semicircle laws and off-diagonal laws defined by two-periodic Jacobi sequences which becomes semicircle laws if the second moments of $\omega_{p,q}(u)$ and $\omega_{q,p}(u)$ are equal.

In this general setting, we prove that the ensemble of symmetric blocks of independent Hermitian Gaussian random matrices

$$\{T_{p,q}(u, n) : p, q \in [r], u \in \mathcal{U}\}$$

converges in moments as $n \rightarrow \infty$ to the ensemble of symmetrized Gaussian operators under the normalized partial traces, namely

$$\lim_{n \rightarrow \infty} \tau_q(n)(T_{p_1,q_1}(u_1, n) \dots T_{p_m,q_m}(u_m, n)) = \Psi_q(\hat{\omega}_{p_1,q_1}(u_1) \dots \hat{\omega}_{p_m,q_m}(u_m)),$$

where $u_1, \dots, u_m \in \mathcal{U}$, all remaining indices belong to $[r] := \{1, 2, \dots, r\}$, and $\tau_q(n)$ denotes the *normalized partial trace* over the set of basis vectors $\{e_k : k \in N_q\}$ composed with classical expectation (called a *partial trace* in the sequel).

This result can be viewed as a block refinement of that used by Voiculescu in his fundamental asymptotic freeness result [32]. Using the convex linear combination

$$\Psi = \sum_{q=1}^r d_q \Psi_q,$$

we easily obtain a similar formula for the normalized trace composed with classical expectation $\tau(n)$ (called the *trace* in the sequel) considered by most authors in their studies of asymptotics of random matrices. It is obvious that asymptotic freeness is a special case of our asymptotic matricial freeness and it corresponds to the case when the variances of all $|Y_{i,j}(u, n)|$ are identical. However, we also show that by considering partial traces we can produce random matrix models for boolean independence, monotone independence and s-freeness. It is not a coincidence since all these notions of independence arise in the context of suitable decompositions of free random variables as shown in [21, 22].

Using the concept of matricial freeness, we studied asymptotic distributions of symmetric blocks of one Hermitian Gaussian random matrix in [23]. In this paper, we generalize that result in two important directions. Firstly, we consider symmetric blocks of the ensemble of independent Hermitian and non-Hermitian Gaussian random matrices. Secondly, we include the case when some elements of the *dimension matrix*

$$D = \text{diag}(d_1, d_2, \dots, d_r),$$

called *asymptotic dimensions*, are equal to zero. This corresponds to the situation when at least one dimension of some blocks does not grow fast enough as $n \rightarrow \infty$. Of special interest are off-diagonal symmetric blocks consisting of two rectangular blocks whose dimensions do not grow proportionately, with the larger dimension proportional to n and the smaller dimension growing slower than n (such blocks will be called *unbalanced*). In the formulas for joint distributions under the trace $\tau(n)$, the contributions from partial traces associated with vanishing asymptotic dimensions disappear in the limit. However, certain limit joint distributions involving unbalanced symmetric blocks under the partial traces themselves become non-trivial and interesting.

Limit distributions of the sums $Y(n)$ of independent Hermitian Gaussian random matrices can be described in terms of convolutions of arrays of matricial semicircle laws. Namely, one can define the convolution of arrays of distributions, where the diagonal distributions and the off-diagonal ones are convolved by means of free and boolean convolutions, respectively. Thus

$$[\sigma] = [\sigma(1)] \boxplus [\sigma(2)] \boxplus \dots \boxplus [\sigma(m)]$$

is a convolution of matricial semicircle laws, where the notation \boxplus is used since this convolution is a generalization of the free additive convolution to the category of arrays of distributions. In turn, the limit distributions of $Y(n)$ under the trace are obtained by taking the *matricially free convolution* \boxtimes of the entries of $[\sigma]$. This convolution is introduced and studied in this paper.

The convolution \boxplus is related to the *strongly matricially free convolution* introduced and studied in [24]. We can say that it describes the asymptotic relation between symmetric blocks having the same position in the considered independent matrices $Y(u, n)$, whereas the convolution \boxtimes describes the asymptotic relation between symmetric blocks having different positions in the matrix $Y(n)$. Putting two convolutions together, we obtain a formula called the Double Convolution Formula.

Next, we derive the limit joint distributions of the ensemble of symmetric blocks of independent non-Hermitian Gaussian random matrices as it was done by Voiculescu in the case of independent non-Hermitian Gaussian random matrices themselves [32]. Further, by considering off-diagonal symmetric blocks, we can recover limit joint distributions of independent Wishart matrices

$$W(u, n) = B(u, n)B^*(u, n)$$

where each $B(u, n)$ is a Gaussian random matrix [37], as well as distributions of matrices of similar form $W(n) = B(n)B^*(n)$, where $B(n)$ is a sum or a product of independent Gaussian random matrices. In all these situations, it suffices to embed the considered matrices in the algebra of symmetric blocks in an appropriate way. In particular, we obtain the random matrix model for the noncommutative Bessel laws of Banica *et al* [4]. Non-Hermitian Wishart matrices, studied recently by Kanzieper and Singh [19], can be

treated in a similar way. We also expect that a continuous generalization of our block method should allow us to treat symmetric triangular random matrices studied by Basu *et al* [5] and the triangular ones studied by Dykema and Haagerup [13]. An extension of our results to the class of non-Gaussian random matrices (under suitable assumptions, like uniform boundedness of type given in [12]) seems to present no difficulty.

The paper is organized as follows. Section 2 is devoted to the concept of matricial freeness, the corresponding arrays of matricially free Gaussian operators and matricial semicircle laws. Symmetrized counterparts of these objects are discussed in Section 3. In Section 4, we describe the combinatorics of the mixed moments of operators of both types. The first main result of this paper is contained in Section 5, where we derive the limit joint distributions of the Gaussian Symmetric Block Ensemble under partial traces. The corresponding convolution limit theorem with the Double Convolution Formula is proved in Section 6. In Section 7, we apply this formalism to construct random matrix models for asymptotic boolean independence, monotone independence and s-freeness. In Section 8, we study the transforms of limit distributions and introduce matrix-valued analogs of Catalan numbers. Non-Hermitian Gaussian random matrices are treated in Section 9. Section 10 is devoted to Wishart matrices, matrices of Wishart type and products of independent Gaussian random matrices.

Finally, we take this opportunity to issue an erratum to the definition of the symmetrically matricially free array of units given in [23]. The new definition entails certain changes which are discussed in the Appendix.

2. MATRICIAL SEMICIRCLE LAWS

Let us recall the basic notions related to the concept of matricial freeness [22]. Let \mathcal{A} be a unital $*$ -algebra with an array $(\mathcal{A}_{i,j})$ of non-unital $*$ -subalgebras of \mathcal{A} and let $(\varphi_{i,j})$ be an array of states on \mathcal{A} . Here, and in the sequel, we shall skip \mathcal{J} in the notations involving arrays and we shall tacitly assume that $(i,j) \in \mathcal{J}$, where

$$\mathcal{J} \subseteq [r] \times [r],$$

and r is a natural number. Further, we assume that each $\mathcal{A}_{i,j}$ has an *internal unit* $1_{i,j}$, by which we understand a projection for which $a1_{i,j} = 1_{i,j}a = a$ for any $a \in \mathcal{A}_{i,j}$, and that the unital subalgebra \mathcal{I} of \mathcal{A} generated by all internal units is commutative.

For each natural number m , let us distinguish the subset of the m -fold Cartesian product $\mathcal{J} \times \mathcal{J} \times \dots \times \mathcal{J}$ of the form

$$\mathcal{J}_m = \{((i_1, j_1), \dots, (i_m, j_m)) : (i_1, j_1) \neq \dots \neq (i_m, j_m)\},$$

and its subset

$$\mathcal{K}_m = \{((i_1, i_2), (i_2, i_3), \dots, (i_m, j_m)) : (i_1, i_2) \neq (i_2, i_3) \neq \dots \neq (i_m, j_m)\}.$$

In other words, the neighboring pairs of indices in the set \mathcal{K}_m are not only different (as in free products), but are related to each other as in matrix multiplication. Objects labelled by diagonal or off-diagonal pairs, respectively, will be called *diagonal* and *off-diagonal*.

If $\varphi_1, \varphi_2, \dots, \varphi_r$ are states on \mathcal{A} , we form an array of states $(\varphi_{i,j})$ as follows:

$$\varphi_{i,j} = \varphi_j$$

and then we will say that $(\varphi_{i,j})$ is *defined by the family* (φ_j) . In particular, when $\mathcal{J} = [r] \times [r]$, this array takes the form

$$(\varphi_{i,j}) = \begin{pmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_r \\ \varphi_1 & \varphi_2 & \dots & \varphi_r \\ . & . & \ddots & . \\ \varphi_1 & \varphi_2 & \dots & \varphi_r \end{pmatrix}.$$

The states $\varphi_1, \varphi_2, \dots, \varphi_r$ can be defined as states which are *conjugate* to a distinguished state φ on \mathcal{A} , where conjugation is implemented by certain elements of the diagonal subalgebras, namely

$$\varphi_j(a) = \varphi(b_j^* a b_j)$$

for any $a \in \mathcal{A}$, where $b_j \in \mathcal{A}_{j,j} \cap \text{Ker} \varphi$ is such that $\varphi(b_j^* b_j) = 1$ for any $j \in [r]$.

In general, other arrays can also be used in the definition of matricial freeness, but in this paper we will only use those defined by a family of r states, say (φ_j) , and thus the definition of matricial freeness is adapted to this situation. Since the original definition uses the diagonal states [22], which here coincide with $\varphi_1, \dots, \varphi_r$, we shall use the latter in all definitions.

Definition 2.1. We say that $(1_{i,j})$ is a *matricially free array of units* associated with $(\mathcal{A}_{i,j})$ and (φ_j) if for any state φ_j it holds that

- (a) $\varphi_j(b_1 a b_2) = \varphi_j(b_1) \varphi_j(a) \varphi_j(b_2)$ for any $a \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{I}$,
- (b) $\varphi_j(1_{k,l}) = 1_{j,l}$ for any i, j, k, l ,
- (c) if $a_r \in \mathcal{A}_{i_r, j_r} \cap \text{Ker} \varphi_{i_r, j_r}$, where $1 < r \leq m$, then

$$\varphi_j(a 1_{i_1, j_1} a_2 \dots a_m) = \begin{cases} \varphi_j(a a_2 \dots a_m) & \text{if } ((i_1, j_1), \dots, (i_m, j_m)) \in \mathcal{K}_m \\ 0 & \text{otherwise} \end{cases}.$$

where $a \in \mathcal{A}$ is arbitrary and $((i_1, j_1), \dots, (i_m, j_m)) \in \mathcal{J}_m$.

Definition 2.2. We say that $*$ -subalgebras $(\mathcal{A}_{i,j})$ are *matricially free* with respect to $(\varphi_{i,j})$ if the array of internal units $(1_{i,j})$ is the associated matricially free array of units and

$$\varphi_j(a_1 a_2 \dots a_n) = 0 \quad \text{whenever } a_k \in \mathcal{A}_{i_k, j_k} \cap \text{Ker} \varphi_{i_k, j_k}$$

for any state φ_j , where $((i_1, j_1), \dots, (i_m, j_m)) \in \mathcal{J}_m$. The matricially free array of variables $(a_{i,j})$ in a unital $*$ -algebra \mathcal{A} is defined in a natural way.

The most important example of matricially free operators are the matricial generalizations of free Gaussian operators. They are defined on a suitable Hilbert space of Fock type. In order to define this Fock space, we shall use boolean and free Fock spaces. Recall that by the boolean and free Fock spaces over the Hilbert space \mathcal{H} , respectively, we understand the direct sums

$$\mathcal{F}_0(\mathcal{H}) = \mathbb{C}\Omega \oplus \mathcal{H} \quad \text{and} \quad \mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \mathcal{H}^{\otimes m},$$

where Ω is a unit vector. Both spaces are endowed with the canonical inner products.

Let us assume now that \mathcal{J} contains the diagonal and that \mathcal{U} is a finite index set (in many instances, these assumptions can be dropped). Moreover, in the operatorial context discussed below, we shall use indices (p, q) instead of (i, j) since the related operators give the limit realizations of symmetric blocks $T_{p,q}(u, n)$ built from complex

Gaussian variables $Y_{i,j}(u, n)$ indexed by pairs (i, j) falling into the union of two Cartesian products of intervals N_p and N_q .

Thus, to each $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$ we now associate a Fock space according to the following rule:

$$\mathcal{F}_{p,q}(u) = \begin{cases} \mathcal{F}(\mathbb{C}e_{q,q}(u)) & \text{if } p = q \\ \mathcal{F}_0(\mathbb{C}e_{p,q}(u)) & \text{if } p \neq q \end{cases},$$

where $e_{p,q}(u)$ is a unit vector for any $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$. Each Fock space $\mathcal{F}_{p,q}(u)$ has a distinguished unit (vacuum) vector denoted $\Omega_{p,q}(u)$. In other words, we put free Fock spaces over one-dimensional Hilbert spaces on the diagonal and boolean Fock spaces (over one-dimensional Hilbert spaces) elsewhere. This special type of assignment, which appeared in our previous work [23], is related to the difference between multiplication of diagonal and off-diagonal entries of random matrices.

Definition 2.3. By the *matricially free Fock space of tracial type* we will understand the direct sum of Hilbert spaces

$$\mathcal{M} = \bigoplus_{q=1}^r \mathcal{M}_q,$$

where each summand is of the form

$$\mathcal{M}_q = \mathbb{C}\Omega_q \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{(p_1, p_2, u_1) \neq \dots \neq (p_m, q, u_m)} \mathcal{F}_{p_1, p_2}^0(u_1) \otimes \dots \otimes \mathcal{F}_{p_m, q}^0(u_m),$$

where $\mathcal{F}_{p,q}^0(u)$ is the orthocomplement of $\mathbb{C}\Omega_{p,q}(u)$ in $\mathcal{F}_{p,q}(u)$, endowed with the canonical inner products.

Each space \mathcal{M}_q is similar to the matricially free-boolean Fock space introduced in [23] except that its tensor products do not have to end with diagonal free Fock spaces and that we use a family of arrays of Fock spaces

$$\{\mathcal{F}_{p,q}(u) : (p, q) \in \mathcal{J}, u \in \mathcal{U}\},$$

as well as a family of vacuum vectors

$$\Omega = \{\Omega_1, \dots, \Omega_r\},$$

instead of one array and one distinguished vacuum vector. Of course, each $\Omega_{p,q}(u)$ is different from any vacuum vector from the set Ω . The states defined by $\Omega_1, \dots, \Omega_r$ will be denoted Ψ_1, \dots, Ψ_r , respectively. Thus,

$$\Psi_q(a) = \langle a\Omega_q, \Omega_q \rangle$$

for any $a \in B(\mathcal{M})$ and any $q \in [r]$. By $(\Psi_{p,q})$ we shall denote the array defined by the family (Ψ_q) , thus $\Psi_{p,q} = \Psi_q$ for any $(p, q) \in \mathcal{J}$.

There exists a canonical embedding of \mathcal{M} in the free Fock space over the Hilbert space with the orthonormal basis $\{e_{p,q}(u) : (p, q) \in \mathcal{J}, u \in \mathcal{U}\}$, except that the single vacuum vector in this free Fock space is replaced by the family Ω , namely

$$\tau : \mathcal{M} \rightarrow \left(\bigoplus_{q=1}^r \mathbb{C}\Omega_q \right) \oplus \mathcal{F}^0 \left(\bigoplus_{(p,q,u) \in \mathcal{J} \times \mathcal{U}} \mathbb{C}e_{p,q}(u) \right).$$

These embeddings will be used in the definition of our matricially free Gaussian operators which remind free Gaussian operators, but they have larger kernels.

Remark 2.1. The orthonormal basis \mathcal{B} of the matricially free Fock space \mathcal{M} is given by vacuum vectors $\Omega_1, \dots, \Omega_r$, and simple tensors of the form

$$e_{p_1, p_2}^{\otimes l_1}(u_1) \otimes e_{p_2, p_3}^{\otimes l_2}(u_2) \otimes \dots \otimes e_{p_m, q}^{\otimes l_m}(u_m)$$

where $(p_1, p_2), \dots, (p_m, q) \in \mathcal{J}$, $u_1, \dots, u_m \in \mathcal{U}$ and $l_1, \dots, l_m, m \in \mathbb{N}$, for which it holds that

$$(p_1, p_2, u_1) \neq (p_2, p_3, u_2) \neq \dots \neq (p_m, q, u_m),$$

with the additional condition that $l_k \in \mathbb{N}$ or $l_k = 1$ if (p_k, p_{k+1}) is diagonal or off-diagonal, respectively, for any k . In particular, the above conditions imply that we allow the triples (p, p, u) and (p, p, u') to be neighbors if $u \neq u'$, as well as triples (p, j, u) and (j, k, u) to be neighbors as long as $(p, j) \neq (j, k)$.

Definition 2.4. Let $(\alpha_{i,j}(u))$ be an array of non-negative real numbers for any $u \in \mathcal{U}$. We associate with each such matrix the *matricially free creation operators*

$$\wp_{p,q}(u) = \alpha_{p,q}(u) \tau^* \ell(e_{p,q}(u)) \tau,$$

where $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$, and where $\ell(e_{p,q}(u))\Omega_s = \delta_{q,s}e_{p,q}(u)$, otherwise $\ell(e_{p,q}(u))$ agrees with the free creation operator associated with the vector $e_{p,q}(u)$. The corresponding sums

$$\omega_{p,q}(u) = \wp_{p,q}(u) + \wp_{p,q}^*(u),$$

where $\wp_{p,q}^*(u)$ is the adjoint of $\wp_{p,q}(u)$ (called a *matricially free annihilation operator*), will be called *matricially free Gaussian operators*. The sum

$$\omega(u) = \sum_{p,q} \omega_{p,q}(u)$$

will be called a *Gaussian pseudomatrix*, with $[\omega(u)]$ denoting the matrix $(\omega_{p,q}(u))$.

Note that the above definition is very similar to that in [23] and for that reason we use similar notations and the same terminology. Strictly speaking, however, the above operators live in a larger Fock space, built by the action of a *collection* of arrays of creation operators, namely

$$\{\wp_{p,q}(u) : (p, q) \in \mathcal{J}, u \in \mathcal{U}\}$$

onto the family of vacuum vectors Ω rather than by the action of one array onto one distinguished vacuum vector. Besides, in contrast to [23], we allow $\alpha_{p,q}(u)$ to vanish, which leads to trivial operators.

Proposition 2.1. *The action of $\wp_{p,q}(u)$ onto the basis \mathcal{B} is given by*

$$\begin{aligned} \wp_{p,q}(u)\Omega_q &= \alpha_{p,q}(u)e_{p,q}(u) \\ \wp_{p,q}(u)(e_{q,s}(t) \otimes w) &= \alpha_{p,q}(u)(e_{p,q}(u) \otimes e_{q,s}(t) \otimes w) \end{aligned}$$

for any $e_{q,s}(t) \otimes w \in \mathcal{B}$, where $(p, q), (q, s) \in \mathcal{J}$ and $u, t \in \mathcal{U}$, and in the remaining cases the action gives zero.

Proof. These formulas are easy consequences of Definition 2.4. ■

In other words, up to a scalar multiplication, $\wp_{p,q}(u)$ is given by tensoring on the left by $e_{p,q}(u)$ restricted to the subspace of \mathcal{M} which is matricially related to (p, q) . The commutation relations for the matricially free creation and annihilation operators generalize those for their free counterparts as we show below.

Example 2.1. If \mathcal{J} consists of one element and $\alpha_{1,1}(u) = 1$ for any $u \in \mathcal{U}$, we can write $\wp_{1,1}(u) = \ell(u)$, $\wp_{1,1}^*(u) = \ell^*(u)$ and $\Omega_1 = \Omega$. Then we obtain the action of free creation operators onto the canonical basis of the free Fock space

$$\ell(u)\Omega = e(u) \quad \text{and} \quad \ell(u)(e(t) \otimes w) = e(u) \otimes e(t) \otimes w$$

and the well-known relation between free creation and annihilation operators

$$\ell^*(u)\ell(t) = \delta_{u,t}$$

for any $u, t \in \mathcal{U}$, of which the matricially free creation and annihilation operators are natural generalizations.

Example 2.2. The moments of each operator $\omega_{p,q}(u)$ will be easily obtained from Proposition 2.3 (cf. [23, Proposition 4.2]). Therefore, let us compute some simple *mixed* moments. First, we take only off-diagonal operators:

$$\begin{aligned} \Psi_q(\omega_{p,q}(u)\omega_{k,p}^4(t)\omega_{p,q}(u)) &= \Psi_q(\wp_{p,q}^*(u)\wp_{k,p}^*(t)\wp_{k,p}(t)\wp_{k,p}^*(t)\wp_{k,p}(t)\wp_{p,q}(u)) \\ &= b_{k,p}^2(t)b_{p,q}(u) \end{aligned}$$

where $p \neq q \neq k \neq p$ and $t, u \in \mathcal{U}$ are arbitrary and where $b_{p,q}(u) = \alpha_{p,q}^2(u)$. If some operators are diagonal, we usually obtain more terms. For instance

$$\begin{aligned} \Psi_q(\omega_{p,q}(u)\omega_{p,p}^4(t)\omega_{p,q}(u)) &= \Psi_q(\wp_{p,q}^*(u)\wp_{p,p}^*(t)\wp_{p,p}(t)\wp_{p,p}^*(t)\wp_{p,p}(t)\wp_{p,q}(u)) \\ &\quad + \Psi_q(\wp_{p,q}^*(u)\wp_{p,p}^*(t)\wp_{p,p}^*(t)\wp_{p,p}(t)\wp_{p,p}(t)\wp_{p,q}(u)) \\ &= 2b_{p,p}^2(t)b_{p,q}(u) \end{aligned}$$

for any $p \neq q$ and any $u, t \in \mathcal{U}$. Let us observe that the pairings between the diagonal annihilation and creation operators and between the off-diagonal ones are the same as those between annihilation and creation operators in the free case and in the boolean case, respectively.

Now, using the fact that \mathcal{U} is finite, we would like to prove matricial freeness of *collective Gaussian operators* of the form

$$\omega_{p,q} = \wp_{p,q} + \wp_{p,q}^*,$$

where

$$\wp_{p,q} = \sum_{u \in \mathcal{U}} \wp_{p,q}(u) \quad \text{and} \quad \wp_{p,q}^* = \sum_{u \in \mathcal{U}} \wp_{p,q}^*(u),$$

with respect to the array $(\Psi_{p,q})$. As in [23, Proposition 4.2], we will prove a more general result on matricial freeness of the array of algebras $(\mathcal{A}_{p,q})$ generated by collective creation, annihilation and unit operators.

For that purpose, let us define a suitable array of collective units $(1_{p,q})$. The easiest way to do it is to say that $1_{p,q}$ is the orthogonal projection onto the subspace onto which the associated collective creation or annihilation operators act non-trivially. However, a more explicit definition will be helpful.

Definition 2.5. By the *array of collective units* we will understand $(1_{p,q})$, where

$$1_{p,q} = r_{p,q} + s_{p,q} \quad \text{for } (p, q) \in \mathcal{J},$$

where $s_{p,q}$ is the orthogonal projection onto

$$\mathcal{N}_{p,q} := \bigoplus_{u \in \mathcal{U}} \text{Ran}(\wp_{p,q}(u)),$$

and $r_{p,q}$ is the orthogonal projection onto

$$\begin{aligned}\mathcal{K}_{q,q} &= \mathbb{C}\Omega_q \oplus \bigoplus_{k \neq q} \mathcal{N}_{q,k}, \\ \mathcal{K}_{p,q} &= \mathbb{C}\Omega_q \oplus \bigoplus_k \mathcal{N}_{q,k},\end{aligned}$$

for any diagonal $(q, q) \in \mathcal{J}$ and off-diagonal $(p, q) \in \mathcal{J}$, respectively.

Proposition 2.2. *The following relations hold:*

$$\wp_{p,q}(u)\wp_{k,l}(t) = 0 \text{ and } \wp_{p,q}^*(u)\wp_{k,l}^*(t) = 0$$

for $q \neq k$ and any $u, t \in \mathcal{U}$. Moreover,

$$\wp_{p,q}^*(u)\wp_{p,q}(u) = \begin{cases} b_{q,q}(u)1_{q,q} & \text{if } p = q \\ b_{p,q}(u)r_{p,q} & \text{if } p \neq q \end{cases}$$

and otherwise $\wp_{p,q}^*(u)\wp_{k,l}(t) = 0$ for any $(p, q, u) \neq (k, l, t) \in \mathcal{J} \times \mathcal{U}$, where $b_{p,q}(u) = \alpha_{p,q}^2(u)$.

Proof. These relations follow from Proposition 2.1. ■

By the semicircle law of radius $2\alpha > 0$ we understand the continuous distribution on the interval $[-2\alpha, 2\alpha]$ with density

$$d\sigma = \sqrt{4\alpha^2 - x^2}/2\pi dx$$

and the Cauchy transform

$$G(z) = \frac{z - \sqrt{z^2 - 4\alpha^2}}{2\alpha^2},$$

where the branch of the square root is chosen so that $\sqrt{z^2 - 4\alpha^2} > 0$ if $z \in \mathbb{R}$ and $z \in (2\alpha, \infty)$. In turn, the discrete distribution

$$\kappa = 1/2(\delta_{-\alpha} + \delta_{\alpha})$$

with the Cauchy transform

$$G(z) = \frac{1}{z - \alpha^2/z}$$

defines the Bernoulli law concentrated at $\pm\alpha$.

Proposition 2.3. *For finite \mathcal{U} and any $(p, q) \in \mathcal{J}$, let*

$$\mathcal{A}_{p,q} = \mathbb{C}\langle \wp_{p,q}, \wp_{p,q}^*, 1_{p,q} \rangle$$

and let $(\Psi_{p,q})$ be the array of states on $B(\mathcal{M})$ defined by the states Ψ_1, \dots, Ψ_r associated with the vacuum vectors $\Omega_1, \dots, \Omega_r$, respectively. Then

- (1) the array $(\mathcal{A}_{p,q})$ is matricially free with respect to $(\Psi_{p,q})$, where the array of units is given by Definition 2.5,
- (2) the $\Psi_{q,q}$ -distribution of non-trivial $\omega_{q,q}(u)$ is the semicircle law of radius $2\alpha_{q,q}(u)$ for any $(q, q) \in \mathcal{J}$ and $u \in \mathcal{U}$,
- (3) the $\Psi_{p,q}$ -distribution of non-trivial $\omega_{p,q}(u)$ is the Bernoulli law concentrated at $\pm\alpha_{p,q}(u)$ for any off-diagonal $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$.

Proof. We shall prove only (1) since (2) and (3) follow from the case of one array in [23, Proposition 4.2]. For an off-diagonal pair $(p, q) \in \mathcal{I}$, we have

$$\wp_{p,q}^* \wp_{p,q} = \sum_{u \in \mathcal{U}} \wp_{p,q}^*(u) \wp_{p,q}(u) = b_{p,q} r_{p,q}$$

where $b_{p,q} = \sum_u b_{p,q}(u)$. In turn, for a diagonal pair $(q, q) \in \mathcal{I}$, we have

$$\wp_{q,q}^* \wp_{q,q} = \sum_{u \in \mathcal{U}} \wp_{q,q}^*(u) \wp_{q,q}(u) = b_{q,q} 1_{q,q}$$

where $b_{q,q} = \sum_{u \in \mathcal{U}} b_{q,q}(u)$. The first relation implies that if $\wp_{p,q}$ is non-trivial, then $r_{p,q}, s_{p,q} \in \mathcal{A}_{p,q}$ whenever $p \neq q$. Thus, any element $a \in \mathcal{A}_{p,q}^0 := \mathcal{A}_{p,q} \cap \text{Ker} \Psi_{p,q}$ is of the form

$$a = \alpha \wp_{p,q} + \beta \wp_{p,q}^* + \gamma s_{p,q}$$

for some constants α, β, γ . In turn, if $\wp_{q,q}$ is non-trivial, any element $a \in \mathcal{A}_{q,q}^0 := \mathcal{A}_{q,q} \cap \text{Ker} \Psi_{q,q}$ is spanned by $1_{q,q}$ and products of the form

$$\wp_{q,q}^{n_1}(u_1) \dots \wp_{q,q}^{n_j}(u_j) \wp_{q,q}^{*m_1}(v_{j+1}) \dots \wp_{q,q}^{*m_k}(v_{j+k}),$$

where $\sum_i n_i + \sum_l m_l > 0$ and $u_1 \neq \dots \neq v_{j+k}$ (one uses commutation relations to obtain this form as in [23, Proposition 4.2]). Both facts allow us to conclude that any element of $\mathcal{A}_{q,q}^0$, when acting onto a simple tensor w , either gives zero or a linear combination of vectors of the form

$$e_{q,q}^{\otimes n_1}(u_1) \otimes \dots \otimes e_{q,q}^{\otimes n_j}(v_j) \otimes w$$

if w begins with some $e_{q,t}(u)$, whereas any element of the off-diagonal $\mathcal{A}_{p,q}^0$, when acting onto a simple tensor w which begins with some $e_{q,t}(u)$, gives a linear combination of vectors of the form $e_{p,q}(t) \otimes w$ for some t and otherwise gives zero. Therefore, a successive action of elements a_1, \dots, a_n satisfying the assumptions of Definition 2.2 either gives zero or we have

$$a_1 \dots a_n \Omega_q \in \mathcal{F}_{p_1, q_1}^0 \otimes \dots \otimes \mathcal{F}_{p_n, q}^0 \perp \mathbb{C} \Omega_q,$$

which implies that

$$\Psi_q(a_1 \dots a_n) = 0$$

for any $q \in \mathcal{I}$, which gives the freeness condition of Definition 2.2. The unit conditions of Definition 2.1 are checked by direct computation. In particular, $1_{p,q}$ leaves invariant any vector from $\mathcal{F}_{p_1, q_1}^0 \otimes \dots \otimes \mathcal{F}_{p_n, q}^0$ whenever $q = p_1$ and kills it otherwise. This proves condition (3) of Definition 2.1. Checking the remaining conditions is straightforward. This completes the proof. \blacksquare

Corollary 2.1. *If $\mathcal{I} = [r] \times [r]$, then the array of distributions of non-trivial $\omega_{p,q}(u)$, $u \in \mathcal{U}$, in the states $(\Psi_{p,q})$ takes the matrix form*

$$[\sigma(u)] = \begin{pmatrix} \sigma_{1,1}(u) & \kappa_{1,2}(u) & \dots & \kappa_{1,r}(u) \\ \kappa_{2,1}(u) & \sigma_{2,2}(u) & \dots & \kappa_{2,r}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{r,1}(u) & \kappa_{r,2}(u) & \dots & \sigma_{r,r}(u) \end{pmatrix}$$

where $\sigma_{q,q}(u)$ is the semicircle law of radius $2\alpha_{q,q}(u)$ for any q and $\kappa_{p,q}(u)$ is the Bernoulli law concentrated at $\pm\alpha_{p,q}(u)$ for any $p \neq q$.

Proof. This is a special case of Proposition 2.3. ■

Each matrix of the above form plays the role of a *matricial semicircle law* which is *standard* if each diagonal law is the semicircle law of radius two and each off-diagonal law is the Bernoulli law concentrated at ± 1 . In fact, the family $\{[\sigma(u)], u \in \mathcal{U}\}$ can be treated as a family of *independent* matricial semicircle laws. This aspect will be discussed in Section 6. In the case when some of the operators $\omega_{p,q}(u)$ are equal to zero, the corresponding measures are replaced by δ_0 .

3. SYMMETRIZED MATRICIAL SEMICIRCLE LAWS

We will show in Section 5 that the limit distributions of symmetric blocks of Hermitian Gaussian random matrices can be expressed in terms of matricially free Gaussian operators in the most general case studied in this paper, including the situation when the dimension matrix D is singular. Nevertheless, if both asymptotic dimensions associated with a given symmetric block $T_{p,q}(u, n)$ are non-zero, its limit realization is not given by $\omega_{p,q}(u)$, but by its symmetrization $\hat{\omega}_{p,q}(u)$.

We will show later that in some cases the symmetrized operators will reduce to matricially free Gaussians. For instance, this is the case when the dimension matrix D is of dimension at least two and has one zero on the diagonal. When we evaluate the limit distribution of certain rectangular symmetric blocks under the associated partial trace, we obtain the case studied by Benaych-Georges [6].

In the case when \mathcal{U} consists of one element, the corresponding array of symmetrized Gaussian operators was defined in [23] and thus we only have to extend that definition to the more general situation studied in this paper.

Definition 3.1. If \mathcal{J} is symmetric, then the *symmetrized creation operators* are operators of the form

$$\hat{\wp}_{p,q}(u) = \begin{cases} \wp_{q,q}(u) & \text{if } p = q \\ \wp_{p,q}(u) + \wp_{q,p}(u) & \text{if } p \neq q \end{cases}$$

for any $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$. Their adjoints $\hat{\wp}_{p,q}^*(u)$ will be called *symmetrized annihilation operators*. The *symmetrized Gaussian operators* are sums of the form

$$\hat{\omega}_{p,q}(u) = \hat{\wp}_{p,q}(u) + \hat{\wp}_{p,q}^*(u)$$

for any $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$. Each matrix $(\hat{\omega}_{p,q}(u))$ will be called a *symmetrized Gaussian matrix*.

We have shown in [23] that one array of symmetrized Gaussian operators gives the asymptotic joint distributions of symmetric random blocks of one Gaussian random matrix if the matrix D is non-singular. In other words, symmetric blocks of a square Hermitian Gaussian random matrix behave as symmetrized Gaussian operators when the size of the matrix goes to infinity. We will generalize this result to the case when we have a family of matrices indexed by a finite set \mathcal{U} and we will allow the matrix D to be singular. The second situation leads to the following definition.

Definition 3.2. If $\omega_{p,q}(u) \neq 0$ and $\omega_{q,p}(u) \neq 0$, then $\hat{\omega}_{p,q}(u)$ will be called *balanced*. If $\omega_{q,p}(u) = 0$ and $\omega_{p,q}(u) \neq 0$ or $\omega_{p,q}(u) = 0$ and $\omega_{q,p}(u) \neq 0$, then $\hat{\omega}_{p,q}(u)$ will be called *unbalanced*. If $\omega_{p,q}(u) = \omega_{q,p}(u) = 0$, then $\hat{\omega}_{p,q}(u) = 0$ will be called *trivial*.

The main advantage of using symmetrized Gaussian operators is that in the case when the corresponding matrices $B(u)$ are symmetric and D is non-singular, we can replace the free-boolean convolution of semicircle and Bernoulli laws by the free convolution of semicircle laws

$$\sigma_{p,q}(1) \boxplus \sigma_{p,q}(2) \boxplus \dots \boxplus \sigma_{p,q}(m)$$

for $p \neq q$ (Lemma 6.2). The disadvantage is that this situation is less general and that is why it is the matricial freeness which remains the main notion of independence in the asymptotic regime. Although it is possible to define the notion of symmetric matricial freeness, it is derived from matricial freeness and is less universal. In fact, its formulation in [23] needs to be corrected in order to characterize the algebra generated by symmetrized Gaussian operators and this modification also contributes to some loss of generality. The erratum to the definition of the symmetrically matricially free array of units is given in the Appendix. Nevertheless, the simplicity of Proposition 3.5 and Corollary 3.1 and the importance of the underlying asymptotic model for random matrices seem to provide convincing arguments to treat this case.

Example 3.1. In order to see the difference in computations of moments between the matricially free Gaussian operators and their symmetrizations, consider the following examples:

$$\begin{aligned} \Psi_1(\omega_{1,2}^4) &= 0 \\ \Psi_2(\omega_{1,2}^4) &= \Psi_2(\wp_{1,2}^* \wp_{1,2} \wp_{1,2}^* \wp_{1,2}) = b_{1,2}^2 \\ \Psi_1(\hat{\omega}_{1,2}^4) &= \Psi_1(\omega_{2,1}^4) + \Psi_1(\omega_{2,1} \omega_{1,2}^2 \omega_{2,1}) \\ &= \Psi_1(\wp_{2,1}^* \wp_{2,1} \wp_{2,1}^* \wp_{2,1}) + \Psi_1(\wp_{2,1}^* \wp_{1,2}^* \wp_{1,2} \wp_{2,1}) = b_{2,1}^2 + b_{1,2} b_{2,1} \\ \Psi_2(\hat{\omega}_{1,2}^4) &= \Psi_2(\omega_{1,2}^4) + \Psi_1(\omega_{1,2} \omega_{2,1}^2 \omega_{1,2}) \\ &= \Psi_1(\wp_{1,2}^* \wp_{1,2} \wp_{1,2}^* \wp_{1,2}) + \Psi_1(\wp_{1,2}^* \wp_{2,1}^* \wp_{2,1} \wp_{1,2}) = b_{1,2}^2 + b_{1,2} b_{2,1} \end{aligned}$$

where, for simplicity, we took $\omega_{p,q}(u) = \omega_{p,q}$ for some fixed u in all formulas. If, for instance, $\omega_{1,2} = 0$, then the third moment reduces to $b_{2,1}^2$ and the remaining three moments vanish.

In order to state the relation between the symmetrized creation and annihilation operators, let us define the *collective symmetrized units* in terms of the array $(1_{p,q})$ of Definition 2.5 as

$$\hat{1}_{p,q} = 1_{p,q} + 1_{q,p} - 1_{p,q} 1_{q,p}$$

respectively, for any $(p, q) \in \mathcal{J}$. The symmetrized units are described in more detail in the propositions given below.

Proposition 3.1. *Let \mathcal{J} be symmetric and let $\mathcal{N}_{p,q}$ be defined as in Definition 2.5 for any $(p, q) \in \mathcal{J}$.*

- (1) *If $(q, q) \in \mathcal{J}$, then $\hat{1}_{q,q}$ is the orthogonal projection onto*

$$\mathbb{C}\Omega_q \oplus \bigoplus_k \mathcal{N}_{q,k}.$$

- (2) *If $(p, q) \in \mathcal{J}$, where $p \neq q$, then $\hat{1}_{p,q}$ is the orthogonal projection onto*

$$\mathbb{C}\Omega_p \oplus \mathbb{C}\Omega_q \oplus \bigoplus_k (\mathcal{N}_{p,k} \oplus \mathcal{N}_{q,k}).$$

Proof. The above formulas follow directly from Definition 2.5. ■

Proposition 3.2. *The collective symmetrized units satisfy the following relations:*

$$\hat{1}_{p,q}\hat{1}_{k,l} = \begin{cases} 0 & \text{if } \{p,q\} \cap \{k,l\} = \emptyset \\ \hat{1}_{q,q} & \text{if } p \neq k \text{ and } q = l \\ \hat{1}_{p,q} & \text{if } (p,q) = (k,l) \end{cases}$$

Proof. These relations are immediate consequences of Proposition 3.1. ■

Proposition 3.3. *The following relations hold for any $u, t \in \mathcal{U}$:*

$$\hat{\wp}_{p,q}(u)\hat{\wp}_{k,l}(t) = 0 \quad \text{and} \quad \hat{\wp}_{p,q}^*(u)\hat{\wp}_{k,l}^*(t) = 0$$

when $\{p,q\} \cap \{k,l\} = \emptyset$. Moreover,

$$\hat{\wp}_{p,q}^*(u)\hat{\wp}_{k,l}(t) = 0$$

when $\{p,q\} \neq \{k,l\}$ or $u \neq t$. Finally, if the matrix $(b_{p,q}(u))$ is symmetric, then

$$\hat{\wp}_{p,q}^*(u)\hat{\wp}_{p,q}(u) = b_{p,q}(u)\hat{1}_{p,q}$$

for any $(p,q) \in \mathcal{J}$.

Proof. These relations follow from Proposition 2.2. ■

In view of the last relation of Proposition 3.3, let us assume now that the matrices $(b_{i,j}(u))$ are symmetric for all $u \in \mathcal{U}$, where \mathcal{U} is finite. We would like to define a symmetrized array of algebras related to the array $(\mathcal{A}_{i,j})$ of Proposition 2.3. For that purpose, we introduce *collective symmetrized creation operators*

$$\hat{\wp}_{p,q} = \sum_{u \in \mathcal{U}} \hat{\wp}_{p,q}(u)$$

and their adjoints $\hat{\wp}_{p,q}^*$ for any $(p,q) \in \mathcal{J}$. In turn, the sums

$$\hat{\omega}_{p,q} = \hat{\wp}_{p,q} + \hat{\wp}_{p,q}^* = \sum_{u \in \mathcal{U}} \hat{\omega}_{p,q}(u)$$

will be called the *collective symmetrized Gaussian operators*.

Proposition 3.4. *If $(b_{p,q}(u))$ is symmetric for any $u \in \mathcal{U}$, then*

$$\hat{\wp}_{p,q}^*\hat{\wp}_{p,q} = b_{p,q}\hat{1}_{p,q}, \quad \hat{\wp}_{p,q}\hat{\wp}_{k,l} = 0 \quad \text{and} \quad \hat{\wp}_{p,q}^*\hat{\wp}_{k,l}^* = 0$$

whenever $\{p,q\} \cap \{k,l\} = \emptyset$, where $b_{p,q} = \sum_{u \in \mathcal{U}} b_{p,q}(u)$.

Proof. The proof follows from Proposition 3.3. For instance,

$$\hat{\wp}_{p,q}^*\hat{\wp}_{p,q} = \sum_{u,t \in \mathcal{U}} \hat{\wp}_{p,q}^*(u)\hat{\wp}_{p,q}(t) = \sum_{u \in \mathcal{U}} b_{p,q}(u)\hat{1}_{p,q} = b_{p,q}\hat{1}_{p,q}$$

The remaining relations are proved in a similar way. ■

Let us discuss the notion of symmetric matricial freeness from an abstract point of view. We assume that the set \mathcal{J} is symmetric, by which we mean that $(j,i) \in \mathcal{J}$ whenever $(i,j) \in \mathcal{J}$. Then we consider a symmetric array of subalgebras $(\mathcal{A}_{i,j})$

of a unital algebra \mathcal{A} and we assume that $(1_{i,j})$ is the associated symmetric array of internal units. By \mathcal{I} we denote the unital algebra generated by the internal units and we assume that it is commutative. Moreover, the array of states $(\varphi_{i,j})$ on \mathcal{A} remains as in the definition of matricial freeness. Since the array $(\mathcal{A}_{i,j})$ is symmetric, we thus associate two states, $\varphi_{i,j}$ and $\varphi_{j,i}$, with each off-diagonal subalgebra $\mathcal{A}_{i,j}$.

Denote by $\hat{\mathcal{J}}$ the set of non-ordered pairs $\{i, j\}$, even in the case when $i = j$. Instead of sets \mathcal{J}_m , we shall use their symmetric counterparts, namely subsets of the m -fold Cartesian product $\hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \dots \times \hat{\mathcal{J}}$ of the form

$$\hat{\mathcal{J}}_m = \{(\{i_1, j_1\}, \dots, \{i_m, j_m\}) : \{i_1, j_1\} \neq \dots \neq \{i_m, j_m\}\},$$

and their subsets

$$\hat{\mathcal{K}}_m = \{(\{i_1, j_1\}, \dots, \{i_m, j_m\}) : \{i_k, j_k\} \cap \{i_{k+1}, j_{k+1}\} \neq \emptyset \text{ for } 1 \leq k \leq m-1\}$$

where $m \in \mathbb{N}$. These sets comprise tuples of non-ordered pairs, in which neighboring pairs are different (as in the case of free products) and are related to each other as in non-trivial multiplication of symmetric blocks of matrices. It will be sometimes convenient to use the abbreviated notation $w_k = \{i_k, j_k\}$.

In order to define the notion of *symmetric matricial freeness*, we shall first define the array of symmetrically matricially free units. This definition will differ from that given in [23] since we have discovered that the conditions on the units need to be strengthened in order that the symmetrized array of Gaussian algebras be symmetrically matricially free as claimed in [23, Proposition 4.2].

Definition 3.3. Let $(\mathcal{A}_{i,j})$ be a symmetric array of subalgebras of \mathcal{A} with a symmetric array of internal units $(1_{i,j})$, and let $(\varphi_{i,j})$ be an array of states on \mathcal{A} defined by the family (φ_j) . If, for some $a \in \mathcal{A}_{i,j}^0$, where $i \neq j$, it holds that

$$\varphi_j(b1_{i,i}a) = \begin{cases} \varphi_j(ba) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases},$$

or

$$\varphi_j(b1_{i,i}a) = \begin{cases} \varphi_j(ba) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

for any $b \in \mathcal{A}_{i,j}^0$, then we will say that a is *odd* or *even*, respectively. The subspaces of $\mathcal{A}_{i,j}^0$ spanned by even and odd elements will be called even and odd, respectively. If each off-diagonal $\mathcal{A}_{i,j}^0$ is a vector space direct sum of an odd subspace and an even subspace, the array $(\mathcal{A}_{i,j})$ will be called *decomposable*.

Example 3.2. The idea of even and odd elements comes from the algebras generated by the symmetrized creation operators. If $p \neq q$ and k is odd, it is easy to see that

$$\begin{aligned} \Psi_q(b\hat{1}_{p,p}\hat{\wp}_{p,q}^k) &= \Psi_q(b\wp_{p,q}\wp_{q,p}\dots\wp_{p,q}) \\ &= \Psi_q(b\hat{\wp}_{p,q}^k) \\ \Psi_q(b\hat{1}_{q,q}\hat{\wp}_{p,q}^k) &= 0 \end{aligned}$$

for any $b \in \mathcal{A}_{p,q}^0$ and thus $\hat{\wp}_{p,q}^k$ is odd. In turn, if k is even,

$$\begin{aligned} \Psi_q(b\hat{1}_{q,q}\hat{\wp}_{p,q}^k) &= \Psi_q(b\wp_{q,p}\wp_{p,q}\dots\wp_{p,q}) \\ &= \Psi_q(b\hat{\wp}_{p,q}^k) \end{aligned}$$

$$\Psi_q(b\hat{1}_{p,p}\hat{\wp}_{p,q}^k) = 0$$

for any $b \in \mathcal{A}_{p,q}^0$ and thus $\hat{\wp}_{p,q}^k$ is even. The main property of the pair $(\wp_{p,q}, \wp_{q,p})$ used here is that only alternating products of these two operators do not vanish and thus a non-trivial product of odd or even order maps Ω_q into the linear span of simple tensors which begin with $e_{p,q}(u)$, or $e_{q,p}(u)$, where $u \in \mathcal{U}$, respectively.

Definition 3.4. We say that the symmetric array $(1_{i,j})$ is a *symmetrically matricially free array of units* associated with a symmetric decomposable array $(\mathcal{A}_{i,j})$ and the array $(\varphi_{i,j})$ if for any state φ_j it holds that

- (1) $\varphi_j(u_1 a u_2) = \varphi_j(u_1) \varphi_j(a) \varphi_j(u_2)$ for any $a \in \mathcal{A}$ and $u_1, u_2 \in \mathcal{I}$,
- (2) if $a_r \in \mathcal{A}_{i_r, j_r} \cap \text{Ker} \varphi_{i_r, j_r}$, where $1 < r \leq m$ and $(w_1, \dots, w_m) \in \hat{\mathcal{K}}_m$, then

$$\varphi_j(a 1_{i_1, j_1} a_2 \dots a_m) = \varphi_j(a a_2 \dots a_m)$$

for any $a \in \mathcal{A}$, whenever one of the following cases holds:

- (a) $w_1 \cap w_2 \neq w_2 \cap w_3$ and a_2 is off-diagonal and odd,
- (b) $w_1 \cap w_2 = w_2 \cap w_3$ and a_2 is diagonal or off-diagonal and even,

where we set $w_3 = \{j, j\}$ for $m = 2$, and for any other $(w_1, \dots, w_m) \in \hat{\mathcal{J}}_m$ the moment vanishes.

Definition 3.5. We say that a symmetric decomposable array $(\mathcal{A}_{i,j})$ is *symmetrically matricially free* with respect to $(\varphi_{i,j})$ if

- (1) for any $a_k \in \text{Ker} \varphi_{i_k, j_k} \cap \mathcal{A}_{i_k, j_k}$, where $k \in [m]$ and $(w_1, \dots, w_m) \in \hat{\mathcal{J}}_m$ and for any state φ_j it holds that

$$\varphi_j(a_1 a_2 \dots a_m) = 0$$

- (2) $(1_{i,j})$ is the associated symmetrically matricially free array of units.

The array of variables $(a_{i,j})$ in a unital algebra \mathcal{A} will be called *symmetrically matricially free* with respect to $(\varphi_{i,j})$ if there exists a symmetrically matricially free array of units $(1_{i,j})$ in \mathcal{A} such that the array of algebras $(\mathcal{A}_{i,j})$, each generated by $a_{i,j} + a_{j,i}$ and $1_{i,j}$, respectively, is symmetrically matricially free with respect to $(\varphi_{i,j})$. The definition of *-symmetrically matricially free arrays of variables is similar to that of *-matricially free arrays.

Proposition 3.5. Assume that \mathcal{J} and $(\alpha_{p,q}(u))$ are symmetric and $\alpha_{p,q}(u) > 0$ for any $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$. Let

$$\hat{\mathcal{A}}_{p,q} = \mathbb{C}\langle \hat{\wp}_{p,q}, \hat{\wp}_{p,q}^* \rangle$$

and let $(\Psi_{p,q})$ be the array of states on $B(\mathcal{M})$ defined by the states Ψ_1, \dots, Ψ_r associated with the vacuum vectors $\Omega_1, \dots, \Omega_r$, respectively. Then

- (1) the array $(\hat{\mathcal{A}}_{p,q})$ is symmetrically matricially free with respect to $(\Psi_{p,q})$, where the associated array of units is $(\hat{1}_{p,q})$,
- (2) the $\Psi_{p,q}$ -distribution of $\hat{\omega}_{p,q}(u)$ is the semicircle law of radius $2\alpha_{p,q}(u)$ for any $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$.

Proof. The proof of (2) is the same as in the case when \mathcal{U} consists of one element [23, Proposition 8.1], so we will only be concerned with proving (1). The proof of that one resembles that of Proposition 2.3. In that connection, let us observe that $\hat{1}_{p,q} \in \hat{\mathcal{A}}_{p,q}$ for any $(p, q) \in \mathcal{J}$ by Proposition 3.4, so that is why we did not include

the symmetrized units in the set of generators. In order to prove symmetric matricial freeness of the array $(\hat{\mathcal{A}}_{p,q})$, one needs to prove that the conditions of Definitions 3.4 and 3.5 are satisfied. If $p \neq q$, in view of commutation relations of Proposition 3.3, any element $a \in \hat{\mathcal{A}}_{p,q}^0 := \hat{\mathcal{A}}_{p,q} \cap \text{Ker} \Psi_{p,q}$ is a linear combination of products of the form

$$\hat{\wp}_{p,q}(u_1) \dots \hat{\wp}_{p,q}(u_i) \hat{\wp}_{p,q}^*(t_1) \dots \hat{\wp}_{p,q}^*(t_j),$$

where $u_1, \dots, u_i, t_1, \dots, t_j \in \mathcal{U}$ and $i + j > 0$. Therefore, any element of $\hat{\mathcal{A}}_{p,q}^0$ of the above form, when acting onto a simple tensor w which does not begin with $e_{p,q}(u)$ or $e_{q,p}(u)$ for any u , either gives zero or

$$e_{p_1,q_1}(u_1) \otimes e_{p_2,q_2}(u_2) \dots \otimes e_{p_i,q_i}(u_i) \otimes w$$

whenever w begins with $e_{q_i,k}(u)$ for some k and u , where all pairs (p_j, q_j) belong to $\{(p, q), (q, p)\}$ and alternate, where $1 \leq j \leq i$. In turn, the result of the action of $\hat{\mathcal{A}}_{p,q}^0$ for $p = q$ is identical to that treated in the proof of Proposition 2.3. Therefore, the action of the product $a_1 \dots a_m$ satisfying the assumptions of Definition 3.5 onto Ω_q either gives zero or a vector which is orthogonal to $\mathbb{C}\Omega_q$, which implies that

$$\Psi_q(a_1 \dots a_m) = 0,$$

which proves the first condition of Definition 3.5. Let us prove now that $(\hat{1}_{p,q})$ satisfies the conditions of Definition 3.4. The first condition holds since an analogous condition holds for the array $(1_{p,q})$. To prove the second condition, first observe that each off-diagonal $\mathcal{A}_{p,q}^0$ is decomposable in the sense of Definition 3.3. Namely, the odd (even) subspace of $\mathcal{A}_{p,q}^0$ is spanned by products of creation and annihilation operators of the form given above in which i is odd (even). Suppose first that

$$\{p_1, q_1\} \cap \{p_2, q_2\} = \{p_2, q_2\} \cap \{p_3, q_3\},$$

with $p_2 \neq q_2$, for instance $q_1 = p_2 = p_3$, where indices p_k, q_k are used instead of i_k, j_k . If $a_2 \in \mathcal{A}_{p_2,q_2}^0$ is even, then $a_2 \dots a_m \Omega_q$ is a linear combination of vectors of the form $e_{p_2,q_2}(u) \otimes w$, where $u \in \mathcal{U}$ and w is a simple tensor, which are left invariant by the unit $\hat{1}_{p_1,q_1}$ by Proposition 3.1. If a_2 is odd, then $a_2 \dots a_m \Omega_q$ is a linear combination of vectors of the form $e_{q_2,p_2}(u) \otimes w$, where $u \in \mathcal{U}$ and w is a simple tensor, which are killed by the unit $\hat{1}_{p_1,q_1}$ since $q_1 = p_2 \neq q_2$. Next, suppose that

$$\{p_1, q_1\} \cap \{p_2, q_2\} \neq \{p_2, q_2\} \cap \{p_3, q_3\},$$

with $p_2 \neq q_2$, for instance $q_1 = p_2$ and $q_2 = p_3$. If a_2 is odd, then the vector $a_2 \dots a_m \Omega_q$ is a linear combination of vectors of the form $e_{p_2,q_2}(u) \otimes w$, where $u \in \mathcal{U}$ and w is a simple tensor, which are left invariant by the unit $\hat{1}_{p_1,q_1}$ by Proposition 3.1. If a_2 is even, then $a_2 \dots a_m \Omega_q$ is a linear combination of vectors of the form $e_{q_2,p_2}(u) \otimes w$, where $u \in \mathcal{U}$ and w is a simple tensor, which are killed by the unit $\hat{1}_{p_1,q_1}$ since $q_1 = p_2 \neq q_2$. Finally, if $p_2 = q_2$ and $p_2 \in \{p_1, q_1\}$, then $\hat{1}_{p_1,q_1}$ leaves $a_2 \dots a_m \Omega_q$ invariant. This completes the proof of symmetric matricial freeness. \blacksquare

Corollary 3.1. *Under the assumptions of Proposition 3.5, if $\mathcal{J} = [r] \times [r]$, then the array of distributions of $(\hat{\omega}_{p,q}(u))$, $u \in \mathcal{U}$, in the states $(\Psi_{p,q})$ is given by*

$$[\hat{\sigma}(u)] = \begin{pmatrix} \sigma_{1,1}(u) & \sigma_{1,2}(u) & \dots & \sigma_{1,r}(u) \\ \sigma_{1,2}(u) & \sigma_{2,2}(u) & \dots & \sigma_{2,r}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,r}(u) & \sigma_{2,r}(u) & \dots & \sigma_{r,r}(u) \end{pmatrix}$$

where $\sigma_{p,q}(u)$ is the semicircle law of radius $2\alpha_{p,q}(u)$ for any $(p,q) \in \mathcal{J}$ and $u \in \mathcal{U}$.

Each symmetric matrix of the above form plays the role of a *symmetrized matricial semicircle law*. It clearly differs from the matricial semicircle law of Corollary 2.1, but it contains all information needed to compute all mixed moments of the symmetrized Gaussian operators in the states Ψ_1, \dots, Ψ_r . For that reason, we use square matrices since we need two distributions of each off-diagonal operator $\hat{\omega}_{p,q}(u)$, in the states Ψ_p and Ψ_q .

Remark 3.1. In the case when not all matrices $A(u) = (\alpha_{p,q}(u))$ are symmetric, symmetric matricial freeness is no longer sufficient due to more complicated relations between the collective creation and annihilation operators, namely

$$\hat{\sigma}_{p,q}^* \hat{\sigma}_{p,q} = b_{p,q} r_{p,q} + b_{q,p} r_{q,p}$$

where $b_{p,q} = \sum_{u \in \mathcal{U}} b_{p,q}(u)$ and $b_{q,p} = \sum_{u \in \mathcal{U}} b_{q,p}(u)$ for any $p \neq q$. Moreover, we have relations

$$r_{p,q} \hat{\sigma}_{p,q} = \wp_{q,p}, \quad \hat{\sigma}_{p,q} r_{p,q} = \wp_{p,q}, \quad \wp_{p,q}^* \wp_{p,q} = b_{p,q} r_{p,q}, \quad \hat{1}_{p,q} = r_{p,q} + r_{q,p}$$

and thus, assuming that $b_{p,q} \neq 0$ and $b_{q,p} \neq 0$, we can see that the $*$ -algebra $\hat{\mathcal{A}}_{p,q}$ generated by $\hat{\sigma}_{p,q}$ coincides with the $*$ -algebra generated by $\wp_{p,q}, \wp_{q,p}, r_{p,q}, r_{q,p}$ and thus it contains $\hat{1}_{p,q}$. One can show that the mixed moments of interest, like those computed in Proposition 4.3, satisfy the same recurrences as those in the definition of symmetric matricial freeness. However, there are problems with mixed moments containing variables like $r_{p,q}$ and thus, on the level of algebras, we are not able to fit the array $(\hat{\mathcal{A}}_{p,q})$ into the scheme of symmetric matricial freeness. This might indicate that from the point of view of independence the scheme of matricial freeness is the most natural one in this case.

Let us only remark in this context that in this non-symmetric case the distributions of the operators $\hat{\omega}_{p,q}(u)$ in the states $\Psi_{p,q}$ can be easily determined. For that purpose, we shall need the probability measure ϑ on \mathbb{R} corresponding to the two-periodic continued fraction (a, b, a, \dots) . Its Cauchy transform is

$$G(z) = \frac{z^2 + b - a - \sqrt{(z^2 - b - a)^2 - 4ab}}{2zb}.$$

It has the absolutely continuous part

$$d\vartheta = \frac{\sqrt{4ab - (x^2 - a - b)^2}}{2\pi bx} dx$$

supported on $|\sqrt{a} - \sqrt{b}| \leq |x| \leq \sqrt{a} + \sqrt{b}$ and an atom of mass $1/2 - a/2b$ at $x = 0$ if $a \neq b$ [21, Example 9.2]. In particular, if $a = b$, then ϑ is the semicircle law of radius $2a$.

Corollary 3.2. *If \mathcal{J} is symmetric and $\alpha_{p,q}(u) > 0$ for any $(p, q) \in \mathcal{J} = [r] \times [r]$ and $u \in \mathcal{U}$, then the array of distributions of $(\hat{\omega}_{p,q}(u))$, $u \in \mathcal{U}$, in the states $(\Psi_{p,q})$ is given by*

$$[\hat{\sigma}(u)] = \begin{pmatrix} \sigma_{1,1}(u) & \vartheta_{1,2}(u) & \dots & \vartheta_{1,r}(u) \\ \vartheta_{2,1}(u) & \sigma_{2,2}(u) & \dots & \vartheta_{2,r}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{r,1}(u) & \vartheta_{r,2}(u) & \dots & \sigma_{r,r}(u) \end{pmatrix}$$

where and $\vartheta_{p,q}(u)$ is the distribution corresponding to the two-periodic continued fraction with the Jacobi sequence $(b_{p,q}(u), b_{q,p}(u), \dots)$ for any $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$.

Proof. The diagonal distributions follows from Corollary 2.1, whereas the off-diagonal ones will be computed in Example 5.2. \blacksquare

Each array $[\hat{\sigma}(u)]$ is the symmetrized counterpart of $[\sigma(u)]$ and is called a *symmetrized matricial semicircle law*. Both arrays uniquely determine the family of limit distributions of blocks of independent Hermitian Gaussian random matrices under the partial traces and thus under the trace. Moreover, there is a relation between their off-diagonal distributions of the form

$$\vartheta_{p,q}(u) = \kappa_{p,q}(u) \boxplus \kappa_{q,p}(u),$$

where \boxplus is the s-free convolution defined by the subordination functions [21]. However, both arrays correspond to different matricial convolutions and require independent computations, although they are both based on matricial freeness in one way or another. In turn, the arrays $(\omega_{p,q}(u))$ and $(\hat{\omega}_{p,q}(u))$ are different matricial generalizations of free semicircular families. Further, in the case when $\omega_{p,q}(u) = 0$, the corresponding measures $\sigma_{p,q}(u)$, $\kappa_{p,q}(u)$ and $\vartheta_{p,q}(u)$ should be replaced by δ_0 .

4. COMBINATORICS OF MIXED MOMENTS

It can be seen from the computations in Section 2 that the mixed moments of matrixially free Gaussian operators are related to non-crossing pair partitions. The difference between them and free Gaussians is that one has to use colored non-crossing pair partitions, where coloring is adapted to the pairs of matricial indices from \mathcal{J} and to the additional indices from the set \mathcal{U} . It will be convenient to assume here that $\mathcal{U} = [t]$, where t is a natural number.

For a given non-crossing pair partition π , we denote by $\mathcal{B}(\pi)$, $\mathcal{L}(\pi)$ and $\mathcal{R}(\pi)$ the sets of its blocks, their left and right legs, respectively. If $\pi_i = \{l(i), r(i)\}$ and $\pi_j = \{l(j), r(j)\}$ are blocks of π with left legs $l(i)$ and $l(j)$ and right legs $r(i)$ and $r(j)$, respectively, then π_i is *inner* with respect to π_j if $l(j) < l(i) < r(i) < r(j)$. In that case π_j is *outer* with respect to π_i . It is the *nearest outer block* of π_i if there is no block $\pi_k = \{l(k), r(k)\}$ such that $l(j) < l(k) < l(i) < r(i) < r(k) < r(j)$. Since the nearest outer block, if it exists, is unique, we can write in this case

$$\pi_j = o(\pi_i), \quad l(j) = o(l(i)) \text{ and } r(j) = o(r(i)).$$

If π_i does not have an outer block, it is called a *covering* block. In that case we set $o(\pi_i) = \pi_0$, where we define $\pi_0 = \{0, m+1\}$ and call the *imaginary block*.

Example 4.1. Let $\sigma \in \mathcal{NC}_6^2$ be as in Fig. 1. Its blocks are $\{1, 6\}, \{2, 3\}, \{4, 5\}$ and the imaginary block is $\{0, 7\}$. The left and right legs of π are $\mathcal{L}(\sigma) = \{1, 2, 4\}$ and $\mathcal{R}(\sigma) = \{3, 5, 6\}$. The block $\{1, 6\}$ is the nearest outer block of both $\{2, 3\}$ and $\{4, 5\}$ and the imaginary block $\{0, 7\}$ is the nearest outer block of $\{1, 6\}$.

Computations of mixed moments of $(\omega_{p,q}(u))$ in the states Ψ_q are based on the classes of colored non-crossing pair partitions adapted to ordered tuples of indices. Since, in addition to the pair of matricial indices (p, q) , we have an additional index u as compared with the case studied previously, we distinguish the matricial indices from the non-matricial one and we use an abbreviated notation for the pair $v = (p, q)$. If the set \mathcal{U} consists of one element, we recover the analogous definition of [23].

Definition 4.1. We will say that $\pi \in \mathcal{NC}_m^2$ is *adapted* to the tuple $((v_1, u_1), \dots, (v_m, u_m))$, where $v_k = (p_k, q_k) \in [r] \times [r]$ and $u_k \in [t]$ for any k , if

- (a) $(v_i, u_i) = (v_j, u_j)$ whenever $\{i, j\}$ is a block of π ,
- (b) $q_j = p_{o(j)}$ whenever $\{i, j\}$ is a block of π which has an outer block.

The set of such partitions will be denoted by $\mathcal{NC}_m^2((v_1, u_1), \dots, (v_m, u_m))$. In turn, by $\mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))$ we will denote its subset, for which $q_k = q$ whenever k belongs to a covering block.

In order to find combinatorial formulas for the moments of our Gaussian operators, we need to use colored non-crossing pair partitions. It will suffice to color each $\pi \in \mathcal{NC}_m^2$, where m is even, by numbers from the set $[r]$. We will denote by $F_r(\pi)$ the set of all mappings from the set of blocks of π into $[r]$ called *colorings*. By a *colored non-crossing partition* we will understand a pair (π, f) , where $\pi \in \mathcal{NC}_m$ and $f \in F_r(\pi)$. The set

$$\mathcal{B}(\pi, f) = \{(\pi_1, f), \dots, (\pi_k, f)\}$$

will denote the set of its blocks. We will always assume that also the imaginary block is also colored by a number from the set $[r]$ and thus we can speak of a coloring of $\hat{\pi}$.

Example 4.2. If $\pi \in \mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))$, then $((v_1, u_1), \dots, (v_m, u_m))$ defines a unique coloring of π and $\hat{\pi}$ in which the block containing k is colored by p_k for any k and the imaginary block is colored by q . Consider the first colored partition given in Fig. 1, where, for simplicity, we assume that $t = 1$ and thus we can skip u 's. If we are given the tuple of pairs

$$(v_1, v_2, v_3, v_4) = ((2, 1), (2, 2), (2, 2), (2, 1))$$

then π is adapted to this tuple. If $q = 1$, then the unique coloring of π defined by the given tuple and the number $q = 1$ is given by f_1 since the imaginary block must be colored by 1 and the colors of blocks $\{1, 4\}$ and $\{2, 3\}$ are obtained from the numbers assigned to their left legs, i.e. $f_1(1) = 2$ and $f_1(2) = 2$.

Definition 4.2. Let a real-valued matrix $B(u) = (b_{i,j}(u)) \in M_r(\mathbb{R})$ be given for any $u \in [t]$. We define a family of real-valued functions b_q , where $q \in [r]$, on the set of colored non-crossing pair-partitions by

$$b_q(\pi, f) = b_q(\pi_1, f) \dots b_q(\pi_k, f)$$

where $\pi \in \mathcal{NC}_m^2$ and $f \in F_r(\pi)$, and where b_q is defined on the set of blocks $\mathcal{B}(\pi, f)$ as

$$b_q(\pi_k, f) = b_{i,j}(u),$$

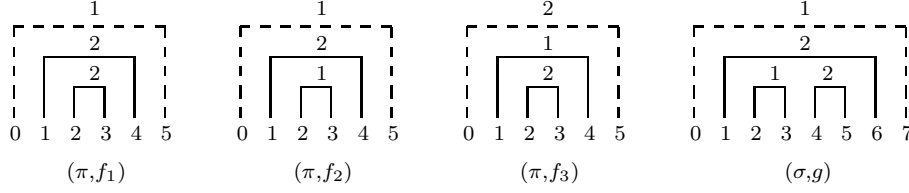


FIGURE 1. Colored non-crossing partitions

whenever block $\pi_k = \{r, s\}$ is colored by i , its nearest outer block is colored by j and $u_r = u_s = u$, where we assume that the imaginary block is colored by q .

It should be remarked that in this paper $b_q(\pi, f)$ may be equal to zero even if $\pi \in \mathcal{NC}_m^2((v_1, u_1), \dots, (v_m, u_m))$ since we assume that the matrices $B(u)$ may contain zeros. Let us also recall our convention saying that if \mathcal{NC}_m^2 or its subset is empty, we shall understand that the summation over $\pi \in \mathcal{NC}_m^2$ or over this subset gives zero. In particular, this will always be the case when m is odd.

Proposition 4.1. *For any tuple $((v_1, u_1), \dots, (v_m, u_m))$ and $q \in [r]$, $m \in \mathbb{N}$, where $v_k = (p_k, q_k) \in [r] \times [r]$ and $u_k \in [t]$ for each k , it holds that*

$$\Psi_q(\omega_{p_1, q_1}(u_1) \dots \omega_{p_m, q_m}(u_m)) = \sum_{\pi \in \mathcal{NC}_{m, q}^2((v_1, u_1), \dots, (v_m, u_m))} b_q(\pi, f)$$

where f is the coloring of π defined by $((v_1, u_1), \dots, (v_m, u_m))$ and the number q .

Proof. The proof is similar to that of [23, Lemma 5.2] and it reduces to showing that if $\epsilon_1, \dots, \epsilon_m \in \{1, *\}$ and $\pi \in \mathcal{NC}_{m, q}^2((v_1, u_1), \dots, (v_m, u_m))$, where m is even, then

$$\Psi_q(\wp_{p_1, q_1}^{\epsilon_1}(u_1) \dots \wp_{p_m, q_m}^{\epsilon_m}(u_m)) = b_q(\pi, f)$$

where f is the coloring of π defined by the collection of indices $\{p_k, k \in \mathcal{L}(\pi)\}$ associated with the left legs of the blocks of π and the index q coloring the imaginary block. The only difference between the former proof and this one is that to each block of (π, f) we assign a matrix element of $B(u)$ for suitable u (u is the same for both legs of each block since the partition satisfies condition (a) of Definition 4.1). ■

Using Proposition 4.1, we can derive nice combinatorial formulas for the moments of sums of collective Gaussian operators

$$\omega = \sum_{p, q} \omega_{p, q}$$

where

$$\omega_{p, q} = \sum_{u \in \mathcal{U}} \omega_{p, q}(u),$$

and for that purpose, denote by $\mathcal{NC}_{m, q}^2[r]$ the set of all non-crossing pair partitions of $[m]$ colored by the set $[r]$, with the imaginary block colored by q .

Lemma 4.1. *The moments of ω in the state Ψ_q , where $q \in [r]$, are given by*

$$\Psi_q(\omega^m) = \sum_{(\pi, f) \in \mathcal{NC}_{m,q}^2[r]} b_q(\pi, f)$$

where $b_q(\pi, f) = b_q(\pi_1, f) \dots b_q(\pi_s, f)$ for $\pi = \{\pi_1, \dots, \pi_s\}$ and

$$b_q(\pi_k, f) = \sum_u b_{i,j}(u)$$

whenever block π_k is colored by i and its nearest outer block is colored by j .

Proof. Using Proposition 4.1, we can express each summand in the formula

$$\Psi_q(\omega^m) = \sum_{p_1, q_1, u_1, \dots, p_m, q_m, u_m} \Psi_q(\omega_{p_1, q_1}(u_1) \dots \omega_{p_m, q_m}(u_m))$$

in terms of $b_q(\pi, f)$, where $\pi \in \mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))$, with $v_k = (p_k, q_k)$ and f is the coloring defined by the tuple $((v_1, u_1), \dots, (v_m, u_m))$. A different choice of $((v_1, u_1), \dots, (v_m, u_m))$ must lead to a different collection of colored partitions (π, f) since even if the same partition appears on the RHS of the formula of Proposition 4.1, the coloring must be different. In fact, π is uniquely determined by the sequence $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ which appears in the nonvanishing moment of type

$$\Psi_q(\wp_{p_1, q_1}^{\epsilon_1}(u_1) \dots \wp_{p_m, q_m}^{\epsilon_m}(u_m)),$$

where $\epsilon_k \in \{1, *\}$. Moments of this type are basic constituents of $\Psi_q(\omega^m)$. For instance, if $\epsilon = (*, *, 1, *, 1, 1)$, the associated unique non-crossing pair partition is σ of Fig.1. If we keep π and change at least one index in the given tuple to which π is adapted, we either get a tuple to which π is not adapted (if we change q_k for some $k \in \mathcal{L}(\pi)$ since this index is determined by the color of its nearest outer block) or we obtain a different coloring f (if we change p_k for some $k \in \mathcal{L}(\pi)$). Therefore, when we add all contributions $b_q(\pi, f)$ of the form given by Proposition 4.1 corresponding to different tuples of indices, we will obtain contributions corresponding to different elements of $\mathcal{NC}_{m,q}^2[r]$. Therefore, in order to prove the formula for $\Psi_q(\omega^m)$, we only need to justify that all colored partitions from $\mathcal{NC}_{m,q}^2[r]$ really do contribute to $\Psi_q(\omega^m)$. Thus, let $(\pi, f) \in \mathcal{NC}_{m,q}^2[r]$ be given. There exists a unique $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ associated with π . In turn, f determines $((p_1, q_1), \dots, (p_m, q_m))$ by an inductive procedure with respect to blocks' depth. Thus, if $\{k, l\}$ is a covering block, we choose $q_k = q_l = q$ and $p_k = p_l = f(k) = f(l)$. Next, if $\{k, l\} = o(\{i, j\})$, then we choose $q_i = q_j = p_k$ and $p_i = p_j = f(i) = f(j)$, etc. We proceed in this fashion until we choose all $p_k, q_k, k \in [m]$. This completes the proof. ■

Example 4.3. For square arrays $(\omega_{i,j}(u))$ of dimension r , where $u \in \mathcal{U}$, and for any $q \in [r]$, we have

$$\begin{aligned} \Psi_q(\omega^4) &= \sum_{i,j,s,t} (\Psi_q(\wp_{i,q}^*(s) \wp_{i,q}(s) \wp_{j,q}^*(t) \wp_{j,q}(t)) + \Psi_q(\wp_{j,q}^*(s) \wp_{i,j}^*(t) \wp_{i,j}(t) \wp_{j,q}(s))) \\ &= \sum_{i,j,s,t} (b_{i,q}(s) b_{j,q}(t) + b_{i,j}(t) b_{j,q}(s)) \\ &= \sum_{f \in F_r(\pi)} b_q(\pi, f) + \sum_{g \in F_r(\sigma)} b_q(\sigma, g) \end{aligned}$$

where $\pi = \{\{1, 2\}, \{3, 4\}\}$ and $\sigma = \{\{1, 4\}, \{2, 3\}\}$ and where $b_q(\pi, f)$ and $b_q(\sigma, f)$ are defined as in the proof of Lemma 4.1.

Note that ω coincides with the sum of all symmetrized Gaussian operators

$$\hat{\omega} = \sum_{p \leq q} \hat{\omega}_{p,q}$$

and thus Lemma 4.1 gives a formula for the moments of $\hat{\omega}$ as well. However, this is not the case for the mixed moments of symmetrized Gaussian operators ($\hat{\omega}_{p,q}(u)$) in the state Ψ_q . These are based on the class of colored non-crossing pair partitions adapted to ordered tuples of indices of type (w, u) , where w is an abbreviated notation for the set $\{p, q\}$.

The definition of this class is given below. Note that it is slightly stronger than that in [23], which is related to the stronger definition of symmetric matricial freeness discussed in Section 3.

Definition 4.3. We say that $\pi \in \mathcal{NC}_m^2$ is *adapted* to the tuple $((w_1, u_1), \dots, (w_m, u_m))$, where $w_k = \{p_k, q_k\}$ and $(p_k, q_k, u_k) \in [r] \times [r] \times [t]$ for any k , if there exists a tuple $((v_1, u_1), \dots, (v_m, u_m))$, where $v_k \in \{(p_k, q_k), (q_k, p_k)\}$ for any k , to which π is adapted. The set of such partitions will be denoted by $\mathcal{NC}_m^2((w_1, u_1), \dots, (w_m, u_m))$. Its subset consisting of those partitions for which $\pi \in \mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))$ will be denoted $\mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$.

Example 4.4. For simplicity, consider the case when $t = 1$ and we can omit u 's. If we are given the tuple of sets

$$(w_1, w_2, w_3, w_4) = (\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}),$$

then the partition π of Fig.1 is adapted to it since there are two tuples,

$$((2, 1), (1, 2), (1, 2), (2, 1)) \text{ and } ((1, 2), (2, 1), (2, 1), (1, 2)),$$

to which π is adapted. If $q = 1$ or $q = 2$, then the associated coloring is given by f_2 or f_3 , respectively. Thus, there are two colored partitions, (π, f_2) and (π, f_3) , associated with the given partition π and the tuple (w_1, w_2, w_3, w_4) . Nevertheless, once we choose the coloring of the imaginary block, the coloring of π is uniquely determined. In turn, if we are given the tuple of sets

$$(w_1, w_2, w_3, w_4, w_5, w_6) = (\{1, 2\}, \{1, 2\}, \{1, 2\}, \{2, 2\}, \{2, 2\}, \{1, 2\})$$

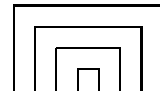
then σ of Fig.1 is adapted to it since it is adapted to the tuple

$$((2, 1), (1, 2), (1, 2), (2, 2), (2, 2), (2, 1)).$$

Here, there is no other tuple of this type to which σ would be adapted and therefore the only coloring associated with π is given by g and the only color which can be assigned to the imaginary block is 1.

Example 4.5. Let us give an example of a partition which satisfies the conditions of Definition 2.3 of [23], but is not adapted in the sense of Definition 4.3 of this paper. Namely, the partition

$$\pi = \{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\} \text{ with diagram}$$



is not adapted to $(\{1, 1\}, \{1, 2\}, \{1, 2\}, \{2, 2\}, \{2, 2\}, \{1, 2\}, \{1, 2\}, \{1, 1\})$ since there is no tuple of ordered pairs (v_1, \dots, v_8) to which π would be adapted according to Definition 4.1 ($\{1, 8\}$ must be colored by 1, this forces $\{2, 7\}$ to be colored by 2 and thus $\{3, 6\}$ to be colored by 1 by condition (b) of Definition 4.1, but $1 \notin \{2, 2\}$). At the same time, π satisfies the conditions of Definition 2.3 of [23]. On the other hand, the partition

$$\sigma = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\} \text{ with diagram } \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}$$

is adapted to $(\{1, 1\}, \{1, 2\}, \{2, 2\}, \{2, 2\}, \{1, 2\}, \{1, 1\})$. The actual reason why σ is adapted to the associated tuple (and π is not) is that the pair $\{1, 2\}$ appears in this tuple an odd (even) number of times in a row and the associated legs belong to different blocks. This feature is related to Definition 3.4, in which even and odd elements appear, and is a consequence of the difference between the action of odd and even positive powers of symmetric off-diagonal blocks $T_{p,q}$ of a matrix onto basis vectors. Namely

$$T_{p,q}^j(V_q) \subseteq V_p \quad \text{and} \quad T_{p,q}^k(V_q) \subseteq V_q$$

for j odd and k even, respectively, where $V_r = \text{span}\{e_i : i \in N_r\}$ and the dependence of all objects involved on n and u is suppressed in the notation.

It can be seen from Example 4.4 that if $\pi \in \mathcal{NC}_m^2((w_1, u_1), \dots, (w_m, u_m))$, then there may be more than one colorings of π defined by the colorings of the associated tuples $((v_1, u_1), \dots, (v_m, u_m))$. Therefore, these tuples may produce more than one coloring of π defined by the sets of matricial indices. However, when we fix q and require that $\pi \in \mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))$, we obtain a unique coloring of π since we have a unique associated tuple $((v_1, u_1), \dots, (v_m, u_m))$.

Proposition 4.2. *If $\pi \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$, there is only one associated tuple $((v_1, u_1), \dots, (v_m, u_m))$ for which $\pi \in \mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))$.*

Proof. If $\{m-1, m\}$ is a block, then the second index of $v_{m-1} = v_m$ must be q and the imaginary block is also the nearest outer block of the block containing $m-2$. This allows us to treat the partition π' obtained from π by removing the block $\{m-1, m\}$ in the same way, which gives the inductive step for this (same depth) case. If $\{m-1, m\}$ is not a block, then the block containing m is the nearest outer block of that containing $m-1$ and thus the second index of v_{m-1} must be equal to the first index of v_m and the second index of v_m is q . This determines v_m and v_{m-1} and gives the first inductive step for this (growing depth) case. Proceeding in this way, we determine v_1, \dots, v_m in a unique way. \blacksquare

Proposition 4.3. *For any tuple $((w_1, u_1), \dots, (w_m, u_m))$ and $q \in [r]$, where $w_k = \{p_k, q_k\}$ and $p_k, q_k \in [r]$, $u_k \in [t]$ for any k and $m \in \mathbb{N}$, it holds that*

$$\Psi_q(\widehat{\omega}_{p_1, q_1}(u_1) \dots \widehat{\omega}_{p_m, q_m}(u_m)) = \sum_{\pi \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))} b_q(\pi, f)$$

where f is the coloring of π defined by $((w_1, u_1), \dots, (w_m, u_m))$ and by the number q .

Proof. The LHS is a sum of mixed moments of the type computed in Proposition 4.1 with perhaps some p_k 's interchanged with the corresponding q_k 's. These moments are associated with tuples of the form $((v_1, u_1), \dots, (v_m, u_m))$. It follows from the

proof of Proposition 4.1 that with each moment of that type we can associate the set $\mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))$ and each such moment contributes

$$\sum_{\pi \in \mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))} b_q(\pi, f),$$

where $\mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m)) \subseteq \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$. It is clear from Definition 4.3 that

$$\bigcup_{(v_1, \dots, v_m)} \mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m)) = \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m)),$$

where the union is taken over the tuples (v_1, \dots, v_m) which are related to (w_1, \dots, w_m) , i.e. if $w_k = \{p_k, q_k\}$, then $v_k \in \{(p_k, q_k), (q_k, p_k)\}$. Since the sets on the LHS are disjoint by Proposition 4.2, the proof is completed. \blacksquare

Finally, we would like to introduce a subset of $\mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$ consisting of those π for which the coloring f satisfies certain additional conditions. Namely, we would like to distinguish only those colored partitions (π, f) whose blocks are not colored by q . Thus, only the imaginary block is colored by q , which is not a contradiction since the imaginary block is not a block of π . This subset will be denoted by $\mathcal{NCI}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$.

Example 4.6. In Fig.1, $(\pi, f_1) \in \mathcal{NCI}_{4,1}^2(\{1, 2\}, \{2, 2\}, \{2, 2\}, \{1, 2\})$ since $q = 1$ and no blocks of π are colored by 1. In turn, the remaining colored partitions are not of this type since the colors of their imaginary blocks are assigned to other blocks, too.

Proposition 4.4. *For given $q \in [r]$, if the only unbalanced operators in the given moment are of the form $\hat{\omega}_{p,q}(u) = \hat{\omega}_{q,p}(u) = \omega_{p,q}(u)$, where $q \neq p \in [r]$ and $u \in [t]$, then*

$$\Psi_q(\hat{\omega}_{p_1,q_1}(u_1) \dots \hat{\omega}_{p_m,q_m}(u_m)) = \sum_{\pi \in \mathcal{NCI}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))} b_q(\pi, f)$$

where f is the coloring of π defined by $((w_1, u_1), \dots, (w_m, u_m))$ and the number q .

Proof. The difference between the considered mixed moment and that of Proposition 4.3 is that all operators are balanced except those involving the index q . Namely, if $q \in \{p_k, q_k\}$ for some k , then we have $\omega_{p_k,q}(u_k)$ or $\omega_{q_k,q}(u_k)$ instead of the symmetrized Gaussian operator $\hat{\omega}_{p_k,q_k}(u_k)$ for such k and $u_k \in [t]$. Since the first index in these operators is different from q , we have to eliminate from the set $\mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$ those partitions in which q colors any blocks of π since it is always the first index which colors the block. This means that we are left only with the contributions from $\pi \in \mathcal{NCI}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$, which completes the proof. \blacksquare

The formula of Lemma 4.1 holds irrespective of the number of trivial operators in the arrays $(\omega_{p,q}(u))$ except that some of the contributions vanish. We can derive a similar formula in the case when the only unbalanced operators are of the type studied in Proposition 4.4. By $\mathcal{NCI}_{m,q}^2[r]$ we shall denote the subset of $\mathcal{NC}_{m,q}^2[r]$ consisting of those partitions in which no blocks (other than the imaginary block, which formally does not belong to the partition) are colored by q .

Lemma 4.2. *If the only unbalanced operators in the arrays $(\hat{\omega}_{i,j}(u))$ are of the form $\hat{\omega}_{p,q}(u) = \hat{\omega}_{q,p}(u) = \omega_{p,q}(u)$, where $p \neq q \in [r]$ and $u \in [t]$, then*

$$\Psi_q(\hat{\omega}^m) = \sum_{(\pi, f) \in \mathcal{NCI}_{m,q}^2[r]} b_q(\pi, f)$$

where $b_q(\pi, f) = b_q(\pi_1, f) \dots b_q(\pi_s, f)$ for $\pi = \{\pi_1, \dots, \pi_s\}$ and $q \in [r]$, $m \in \mathbb{N}$, and

$$b_q(\pi_k, f) = \sum_u b_{i,j}(u)$$

whenever block π_k is colored by i and its nearest outer block is colored by j .

Proof. The proof is similar to that of Lemma 4.1 (Proposition 4.4 is used). ■

Example 4.7. Consider square arrays $(\hat{\omega}_{i,j}(u))$ of dimension r , where $u \in \mathcal{U}$, which have unbalanced operators of the form $\hat{\omega}_{i,q}(u) = \omega_{i,q}(u)$ for any $i \neq q$, where $q \in [r]$ is fixed and $u \in [t]$. We have

$$\hat{\omega} = \sum_{i \leq j} \sum_{u \in \mathcal{U}} \hat{\omega}_{i,j}(u) = \sum_{\substack{i,j \\ i \neq q}} \sum_{u \in \mathcal{U}} \omega_{i,j}(u)$$

and thus

$$\begin{aligned} \Psi_q(\hat{\omega}^4) &= \sum_{\substack{s,t,i,j \\ i \neq q, j \neq q}} (\Psi_q(\wp_{i,q}^*(s) \wp_{i,q}(s) \wp_{j,q}^*(t) \wp_{j,q}(t)) + \Psi_q(\wp_{j,q}^*(s) \wp_{i,j}^*(t) \wp_{i,j}(t) \wp_{j,q}(s))) \\ &= \sum_{\substack{s,t,i,j \\ i \neq q, j \neq q}} (b_{i,q}(s) b_{j,q}(t) + b_{i,j}(t) b_{j,q}(s)) \\ &= \sum_{f \in F_{r,q}(\pi)} b_q(\pi, f) + \sum_{f \in F_{r,q}(\sigma)} b_q(\sigma, f) \end{aligned}$$

where $F_{r,s}(\zeta)$ denotes the set of colorings of ζ by the set $[r] \setminus \{q\}$ and the remaining notations are as in Example 4.3. Note that some of the contributions of Example 4.3 disappear.

5. HERMITIAN GAUSSIAN RANDOM MATRICES

The context for the study of random matrices originated by Voiculescu [32] is the following. Let μ be a probability measure on some measurable space without atoms and let $L = \bigcap_{1 \leq p < \infty} L^p(\mu)$ be endowed with the state expectation \mathbb{E} given by integration with respect to μ . The $*$ -algebra of $n \times n$ random matrices is $M_n(L) = L \otimes M_n(\mathbb{C})$ with the state

$$\tau(n) = \mathbb{E} \otimes \text{tr}(n)$$

where $\text{tr}(n)$ is the normalized trace over the set $\{e_j : j \in [n]\}$ of basis vectors of \mathbb{C}^n .

In order to study the asymptotics of symmetric blocks of random matrices, we partition each set $[n]$ into disjoint non-empty intervals

$$[n] = N_1 \cup \dots \cup N_r, \quad \text{where} \quad \lim_{n \rightarrow \infty} \frac{|N_j|}{n} \rightarrow d_j \geq 0$$

for any j , and we set again $D = \text{diag}(d_1, d_2, \dots, d_r)$ to be the associated diagonal dimension matrix. The numbers d_1, \dots, d_r will be called *asymptotic dimensions*.

Definition 5.1. By a *Hermitian Gaussian random matrix* we shall understand a Hermitian n -dimensional matrix $Y(n) = (Y_{i,j}(n))$ of complex-valued random variables such that

- (1) the real-valued variables

$$\{\operatorname{Re}Y_{i,j}(n), \operatorname{Im}Y_{i,j}(n) : 1 \leq i \leq j \leq n\}$$

are independent Gaussian random variables,

- (2) $\mathbb{E}(Y_{i,j}(n)) = 0$ for any i, j, n ,

- (3) $\mathbb{E}(|Y_{i,j}(n)|^2) = v_{p,q}/n$ for any $(i, j) \in N_p \times N_q$ and n , where $U := (v_{p,q}) \in M_r(\mathbb{R})$.

The asymptotic joint distributions of symmetric blocks of one Hermitian Gaussian random matrix in the case when all asymptotic dimensions are positive have been studied in [23], where we have shown that they have the same limit joint distributions as the symmetrized Gaussian operators. In this paper, we shall generalize this result to the case when we have a whole family of Hermitian Gaussian random matrices

$$\{Y(u, n), u \in \mathcal{U}\}$$

for any $n \in \mathbb{N}$. Moreover, we will allow some asymptotic dimensions to vanish.

Definition 5.2. We will say that the family of Hermitian Gaussian random matrices of Definition 5.1 is *independent* if the variables

$$\{\operatorname{Re}Y_{i,j}(u, n), \operatorname{Im}Y_{i,j}(u, n) : 1 \leq i \leq j \leq n \text{ and } u \in \mathcal{U}\}$$

are independent and

$$\mathbb{E}(|Y_{i,j}(u, n)|^2) = \frac{v_{p,q}(u)}{n}$$

for any $(i, j) \in N_p \times N_q$ and $n \in \mathbb{N}, u \in \mathcal{U}$, i.e. the variances of the complex-valued variables $Y_{i,j}(u, n)$ are identical within each block defined by $N_p \times N_q$ for any $u \in \mathcal{U}$.

As in the case of one matrix, we will study the asymptotic joint distributions of the submatrices of the form

$$T_{p,q}(u, n) = \sum_{(i,j) \in N_{p,q}} Y_{i,j}(u, n) \otimes e_{i,j}(n),$$

where $\{e_{i,j}(n) | 1 \leq i, j \leq n\}$ is a system of matrix units for each n and

$$N_{p,q} = (N_p \times N_q) \cup (N_q \times N_p)$$

for $p, q \in [r]$ and $u \in \mathcal{U}$, called *independent symmetric random blocks*. Non-Gaussian random matrices [12] and the corresponding symmetric blocks can be treated in a similar way, but our proofs rely on the Gaussian case treated in [23, 31].

We would like to find a Hilbert space realization of the limit joint distributions of independent symmetric random blocks under $\tau(n)$ and under

$$\tau_q(n) = \mathbb{E} \otimes \operatorname{tr}_q(n),$$

where $\operatorname{tr}_q(n)$ is the *normalized partial trace* over the set $\{e_j : j \in N_q\}$, where by normalization we understand division by $|N_q|$.

Definition 5.3. The sequence of symmetric random blocks $(T_{p,q}(u, n))_{n \in \mathbb{N}}$, where $(p, q) \in \mathcal{J}$, $u \in \mathcal{U}$ are fixed, will be called *balanced* if $d_p > 0$ and $d_q > 0$, *unbalanced* if $(d_p = 0 \text{ and } d_q > 0)$ or $(d_p > 0 \text{ and } d_q = 0)$ and *evanescent* if $d_p = 0$ and $d_q = 0$ (cf. Definition 3.2).

Example 5.1. Consider the sequence of Hermitian Gaussian random matrices of the block form

$$Y(n) = \begin{pmatrix} S_{1,1}(n) & S_{1,2}(n) \\ S_{1,2}^*(n) & S_{2,2}(n) \end{pmatrix},$$

where both diagonal blocks are Hermitian for any $n \in \mathbb{N}$ (u is omitted for simplicity). Thus, we have three symmetric blocks, two diagonal ones, $T_{1,1}(n) = S_{1,1}(n)$ and $T_{2,2}(n) = S_{2,2}(n)$, and the off-diagonal one that can be identified with the Hermitian matrix

$$T_{1,2}(n) = \begin{pmatrix} 0 & S_{1,2}(n) \\ S_{1,2}^*(n) & 0 \end{pmatrix}.$$

If we suppose that

$$\lim_{n \rightarrow \infty} \frac{|N_1|}{n} = d_1 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{|N_2|}{n} = d_2 > 0,$$

then $(T_{1,1}(n))_{n \in \mathbb{N}}$ is evanescent, $(T_{2,2}(n))_{n \in \mathbb{N}}$ is balanced and $(T_{1,2}(n))_{n \in \mathbb{N}}$ is unbalanced. It will follow from Theorem 5.1 that the moments of such symmetric blocks under $\tau_q(n)$ tend to the moments of trivial, balanced and unbalanced symmetrized Gaussian operators, $\hat{\omega}_{1,1} = 0$, $\hat{\omega}_{2,2} = \omega_{2,2}$ and $\hat{\omega}_{1,2} = \omega_{2,1}$, respectively, under Ψ_q , where $q \in \{1, 2\}$.

In fact, it is easy to predict that if D is singular, then all mixed moments involving evanescent blocks will tend to zero. However, if we compute mixed moments involving unbalanced blocks consisting of pairs of mutually adjoint rectangular blocks whose one dimension grows too slowly, it is less obvious that one can still obtain nonvanishing limit moments under a suitable partial trace.

In the Fock space realization of limit joint distributions of blocks under $\tau(n)$ we will always use the convex linear combination of states Ψ_1, \dots, Ψ_r of the form

$$\Psi = \sum_{q=1}^r d_q \Psi_q,$$

which is a state on $B(\mathcal{M})$, but in the study of the limit joint distributions of blocks under $\tau_q(n)$ we will use the single state Ψ_q . In this case, it will not be convenient to assume that Ψ_q is a special case of Ψ with $d_q = 1$ and the remaining asymptotic dimensions vanishing.

As we have proved in [23], the asymptotic joint distributions of blocks of one random matrix are expressed in terms of the matrix $B = DV$ being the product of the dimension matrix and the symmetric matrix of numbers $V = (v_{p,q})$ related to the variances, namely

$$b_{p,q} = d_p v_{p,q}, \quad \text{where} \quad \frac{v_{p,q}}{n} = \mathbb{E}(|Y_{i,j}(n)|^2)$$

for each complex variable $Y_{i,j}(n)$ in the block $T_{p,q}(n)$ of the matrix $Y(n)$. In the present generalization, when we deal with a family of independent random matrices, the situation is similar and we introduce the family of matrices

$$B(u) = DV(u),$$

where $V(u) = (v_{p,q}(u))$ for any $u \in \mathcal{U}$ and thus $b_{p,q}(u) = d_p v_{p,q}(u)$ for any $p, q \in [r]$ and $u \in \mathcal{U}$. Let us observe that even in the case when the random matrices $Y(u, n)$ are Hermitian, which implies that the matrices $V(u)$ are symmetric, the associated matrices

$B(u)$ are symmetric only if all dimensions are equal. In turn, if all variances are equal to one, then

$$B(u) = \begin{pmatrix} d_1 & d_1 & \dots & d_1 \\ d_2 & d_2 & \dots & d_2 \\ \vdots & \vdots & \ddots & \vdots \\ d_r & d_r & \dots & d_r \end{pmatrix},$$

for any $u \in \mathcal{U}$. Even this simple situation is interesting when we investigate joint distributions of blocks. Of course, when computing asymptotic distributions of matrices $Y(n, u)$, we do not need to split them into blocks and free probability suffices.

Theorem 5.1. *Let $\{Y(u, n) : u \in \mathcal{U}\}$ be a family of independent Hermitian Gaussian random matrices for any $n \in \mathbb{N}$ and let $(T_{p,q}(u, n))$, where $u \in \mathcal{U}$ and $n \in \mathbb{N}$, be arrays of their symmetric blocks. Then*

$$\lim_{n \rightarrow \infty} \tau_q(n)(T_{p_1, q_1}(u_1, n) \dots T_{p_m, q_m}(u_m, n)) = \Psi_q(\hat{\omega}_{p_1, q_1}(u_1) \dots \hat{\omega}_{p_m, q_m}(u_m))$$

for any $(p_1, q_1), \dots, (p_m, q_m) \in \mathcal{J}$, $q \in [r]$, and $u_1, \dots, u_m \in \mathcal{U}$, where $\hat{\omega}_{p,q}(u)$'s are symmetrized Gaussian operators associated with matrices $B(u)$.

Proof. We would like to generalize the proof of [23, Theorem 9.1]. First, consider the partial trace $\tau_q(n)(T_{p_1, q_1}(u_1, n) \dots T_{p_m, q_m}(u_m, n))$ for any $q \in [r]$ and even $m = 2s$. We need to enumerate contributions from moments of the form

$$\mathbb{E}(Y_{i_1, i_2}(u_1, n) \dots Y_{i_m, i_1}(u_m, n))$$

which are associated with $\pi(\gamma) \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$, where $w_i = \{p_i, q_i\}$ for $i \in [m]$, defined by permutations $\gamma : [m] \rightarrow [m]$ such that

$$\gamma^2 = \text{id}, \quad \gamma(k) \neq k \quad \text{and} \quad i_{\gamma(k)} = j_k, \quad j_{\gamma(k)} = i_k, \quad u_{\gamma(k)} = u_k$$

Each such moment is $O(n^{-s-1})$ as $n \rightarrow \infty$. For convenience, we restrict our attention to the case when

$$(i_1, j_1) \in N_{p_1} \times N_{q_1}, \dots, (i_m, j_m) \in N_{p_m} \times N_{q_m},$$

since the remaining cases can be treated in a similar way, with some p_i 's interchanged with the corresponding q_i 's. Moreover, we must have $q = p_1$ since the action of $Y_{i_m, i_1}(u_m, n)$ onto vectors e_j , where $i_1 \notin N_q$ and $j \in N_q$, will give zero. Assuming therefore that $q = p_1$, we can see that the number of non-zero moments of the considered type is $\sum_{\gamma} \Theta_n(\gamma)$, where γ runs over the set of permutations of $[m]$ such that

$$\gamma^2 = \text{id}, \quad \gamma(k) \neq k \quad \text{and} \quad q_{\gamma(k)} = p_k, \quad p_{\gamma(k)} = q_k, \quad u_{\gamma(k)} = u_k$$

for any $k \in [m]$. At this stage of the proof, let us observe that the main difference between this proof and that in [23] is that the set of permutations of $[m]$ just described may be smaller since these must be adapted to u_1, \dots, u_m , which, in general, leads to a smaller class of partitions, namely

$$\mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m)) \subseteq \mathcal{NC}_{m,q}^2(w_1, \dots, w_m)$$

for any u_1, \dots, u_m . Here, one has to take into account the erratum given in the Appendix which results from the stronger definition of partitions adapted to the tuples (w_1, \dots, w_m) (Definition 4.3). Thus, only if $\pi(\gamma) \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$,

the number of independent indices which enter in the computation of $\Theta_n(\gamma)$ is $s + 1$. However, the enumeration of the cardinalities of $\Theta_n(\gamma)$ for such γ remains the same:

$$\Theta_n(\gamma) = \text{card}\{(i_1, i_{r(1)}, \dots, i_{r(s)}) \in N_{p_1} \times N_{p_{r(1)}} \times \dots \times N_{p_{r(s)}}\},$$

where

$$\mathcal{R}(\pi(\gamma)) = \{r(1), r(2), \dots, r(s)\}$$

is the set of right legs of $\pi(\gamma)$. Thus the dimension formula for $\Theta_n(\gamma)$ is again of the form

$$\Theta_n(\gamma) = n_{p_1} n_{p_{r(1)}} \dots n_{p_{r(s)}},$$

where $n_p = |N_p|$ for any p . For those independent indices which are pairwise different and correspond to partitions $\pi(\gamma) \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$, the corresponding expectation of complex Gaussian random variables $\mathbb{E}(Y_{i_1, j_1}(u_1, n) \dots Y_{i_m, j_m}(u_m, n))$ is the product of variances indexed by matricial indices and labelled by elements of \mathcal{U} , namely

$$\mathbb{E}(Y_{i_k, j_k}(u_k, n) Y_{i_{\gamma(k)}, j_{\gamma(k)}}(u, n_{\gamma(k)})) = \frac{v_{p_k, q_k}(u_k)}{n}$$

where $\gamma(k) \in \mathcal{R}(\pi(\gamma))$, with $i_{\gamma(k)} = j_k$, $j_{\gamma(k)} = i_k$ and $u_{\gamma(k)} = u_k$, and where the assumption that variances are equal within blocks is used. Let us conclude that in the computations of mixed moments of symmetric blocks taken from a family of independent matrices rather than from one matrix, we just have to replace the class of partitions $\mathcal{NC}_{m,q}^2(w_1, \dots, w_m)$ by $\mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$ and the variances $v_{p,q}$ by $v_{p,q}(u)$.

We now collect expectations corresponding to all $\pi \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$ and take into account that

$$\lim_{n \rightarrow \infty} \frac{\Theta_n(\gamma)}{n_q n^s} = d_{p_{r(1)}} \dots d_{p_{r(s)}},$$

where we use the fact that $p_1 = q$ and thus $n_{p_1}/n_q = 1$. Suppose first that all symmetric blocks in the considered moment are balanced. Then $d_k > 0$ for all relevant k 's, which implies that the contribution from each expectation corresponding to the given π is $b_q(\pi, f)$, where f is the coloring of π defined by q and the tuple (w_1, \dots, w_m) . Of course, the values of $b_q(\pi, f)$ depend on the matrices $B(u_1), \dots, B(u_m)$. Thus, we obtain

$$\sum_{\pi \in \mathcal{NC}_{m,q}^2((v_1, u_1), \dots, (v_m, u_m))} b_q(\pi, f)$$

as $n \rightarrow \infty$. Since this expression is equal to

$$\Psi_q(\hat{\omega}_{p_1, q_1}(u_1) \dots \hat{\omega}_{p_m, q_m}(u_m))$$

by Proposition 4.3, our assertion is proved. In turn, if there is an unbalanced block, then the product of asymptotic dimensions given above is non-zero if the associated vanishing dimension does not label any inner blocks of π . This is possible if and only if $\pi \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$. In this case, we obtain

$$\sum_{\pi \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))} b_q(\pi, f),$$

which, by Proposition 4.4, is equal to $\Psi_q(\hat{\omega}_{p_1, q_1}(u_1) \dots \hat{\omega}_{p_m, q_m}(u_m))$. Finally, if the investigated moment contains one evanescent block, then in each product of asymptotic dimensions $d_{p_{r(1)}} \dots d_{p_{r(s)}}$ at least one dimension vanishes and thus the limit moment is

zero, which agrees with the corresponding moment of symmetrized Gaussian operators since one of them is trivial. This completes the proof. \blacksquare

Corollary 5.1. *Under the assumptions of Theorem 5.1,*

$$\lim_{n \rightarrow \infty} \tau(n)(T_{p_1, q_1}(u_1, n) \dots T_{p_m, q_m}(u_m, n)) = \Psi(\hat{\omega}_{p_1, q_1}(u_1) \dots \hat{\omega}_{p_m, q_m}(u_m))$$

for any $(p_1, q_1), \dots, (p_m, q_m) \in \mathcal{J}$ and $u_1, \dots, u_m \in \mathcal{U}$.

Proof. We have

$$\tau(n) = \sum_{q=1}^r \frac{|N_q|}{n} \tau_q(n),$$

where $|N_q|/n \rightarrow d_q$ as $n \rightarrow \infty$ for any $q \in [r]$, and thus the assertion follows from Theorem 5.1. \blacksquare

Example 5.2. Let $(T_{p,q}(n))_{n \in \mathbb{N}}$ be a sequence of off-diagonal balanced symmetric blocks for fixed $(p, q) \in \mathcal{J}$ ($u \in \mathcal{U}$ is omitted for simplicity). By Theorem 5.1,

$$\lim_{n \rightarrow \infty} \tau_q(n)(T_{p,q}^m(n)) = \Psi_q(\hat{\omega}_{p,q}^m),$$

where we have

$$\Psi_q(\hat{\omega}_{p,q}^m) = \sum_{\epsilon_1, \dots, \epsilon_m \in \{1, *\}} \Psi_q(\hat{\wp}_{p,q}^{\epsilon_1} \dots \hat{\wp}_{p,q}^{\epsilon_m})$$

and only the summands which correspond to tuples $(\epsilon_1, \dots, \epsilon_m)$ which define non-crossing pair partitions give non-zero contributions (cf. [23, Lemma 4.1]), where we skip u in all notations. Moreover, if such a tuple defines a non-crossing pair partition, then all blocks of depth one contribute $b_{p,q}$ since the associated creation operators act on Ω_q , the blocks of depth two contribute $b_{q,p}$ since the associated creation operators act on $e_{p,q}$, blocks of depth three contribute again $b_{p,q}$ since the associated creation operators act on $e_{q,p} \otimes e_{p,q}$, etc. By Proposition 4.3, for even m , we have

$$\Psi_q(\hat{\omega}_{p,q}^m) = \sum_{\pi \in \mathcal{NC}_m^2} b_{p,q}^{|\mathcal{B}_o(\pi)|} b_{q,p}^{|\mathcal{B}_e(\pi)|}$$

since the set of non-crossing pair partitions of $[m]$ which are adapted to the tuple $(\{p, q\}, \dots, \{p, q\})$ of length m and to the number q can be identified with the set of all non-crossing pair partitions of $[m]$, where $\mathcal{B}_o(\pi)$ and $\mathcal{B}_e(\pi)$ denote the blocks of π of odd and even depths, respectively. Therefore, the Cauchy transform of the limit distribution can be represented as the two-periodic continued fraction with alternating Jacobi coefficients $(b_{p,q}, b_{q,p}, b_{p,q}, \dots)$.

Example 5.3. Let $(T_{p,q}(n))_{n \in \mathbb{N}}$ be a sequence of off-diagonal unbalanced symmetric blocks, where $p \neq q$ are fixed and $d_q = 0$ and $d_p \neq 0$ ($u \in \mathcal{U}$ is omitted for simplicity). If m is even, then

$$\lim_{n \rightarrow \infty} \tau_q(n)(T_{p,q}^m(n)) = \Psi_q(\omega_{p,q}^m) = \Psi_q((\wp_{p,q}^* \wp_{p,q})^{m/2}) = b_{p,q}^{m/2}$$

as $n \rightarrow \infty$ since $\omega_{q,p} = 0$. Of course, if m is odd, we get zero. The remaining partial traces vanish, in particular

$$\lim_{n \rightarrow \infty} \tau_p(n)(T_{p,q}^m(n)) = \Psi_p(\omega_{p,q}^m) = 0$$

as $n \rightarrow \infty$, since $\omega_{p,q} \Omega_p = 0$ for $p \neq q$.

It should be clear that if all variances are equal to one, all mixed moments of Theorem 5.1 become polynomials in asymptotic dimensions. This case is interesting especially if not all asymptotic dimensions are equal. Although finding explicit forms of such polynomials may be highly non-trivial, a combinatorial formula can be given.

Definition 5.4. For any tuple $((w_1, u_1), \dots, (w_m, u_m))$, where $w_1 \cup \dots \cup w_m \subseteq [r]$ and $u_1, \dots, u_m \in \mathcal{U}$, with m even, and for any $q \in [r]$, let

$$Q_s(d_1, d_2, \dots, d_r) = \sum_{\pi \in \mathcal{NC}_m^2((w_1, u_1), \dots, (w_m, u_m))} \prod_{j \in w_1 \cup \dots \cup w_m} d_j^{|\mathcal{R}_j(\pi)|}$$

be a polynomial in asymptotic dimensions d_1, \dots, d_r , where $\mathcal{R}_j(\pi)$ is the set of right legs of π which are colored by j .

Remark 5.1. It is not hard to see that all these polynomials are homogenous and can be written in the form

$$Q_s(d_1, d_2, \dots, d_r) = \sum_{\substack{0 \leq k_1, \dots, k_r \leq s \\ k_1 + \dots + k_r = s}} M_{k_1, \dots, k_r} d_1^{k_1} \dots d_r^{k_r}$$

where M_{k_1, \dots, k_r} is the number of those $\pi \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$ which have k_j right legs colored by j , where $j \in [r]$. Clearly, the coefficients M_{k_1, \dots, k_r} depend on the tuple $((w_1, u_1), \dots, (w_r, u_r))$ and on q and may vanish.

Corollary 5.2. Under the assumptions of Theorem 5.1, if $v_{p,q}(u) = 1$ for any $p, q \in [r], u \in \mathcal{U}$, then for any $m = 2s$

$$\lim_{n \rightarrow \infty} \tau_q(n)(T_{p_1, q_1}(u_1, n) \dots T_{p_m, q_m}(u_m, n)) = Q_s(d_1, d_2, \dots, d_r)$$

where $Q_s(d_1, d_2, \dots, d_r)$ is the polynomial of Definition 5.4.

Proof. Non-vanishing contributions correspond to $\pi \in \mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$ and are equal to products of asymptotic dimensions, namely

$$d_{p_r(1)} \dots d_{p_r(s)},$$

where $\mathcal{R}(\pi) = \{r(1), \dots, r(s)\}$, which proves our assertion. ■

In the case when all block variances are equal to one, the family of polynomials of Corollary 5.2 gives all limit mixed moments in our random symmetric block model expressed in terms of asymptotic dimensions. If D is singular, then certain limit mixed moments of Theorem 5.1, and thus also the corresponding polynomials, may vanish. We close this section with giving the necessary and sufficient conditions for them to be non-trivial. This result shows that the situation studied in Lemma 4.2 is essentially the only one which is of interest in the case when at least one symmetric block in the given moment is not balanced.

Corollary 5.3. Under the assumptions of Theorem 5.1,

$$\lim_{n \rightarrow \infty} \tau_q(n)(T_{p_1, q_1}(u_1, n) \dots T_{p_m, q_m}(u_m, n)) \neq 0$$

if and only if the following conditions hold:

- (1) there are no evanescent blocks in the above moment,
- (2) all unbalanced blocks are of the form $T_{p,q}(u, n)$ for some $p \neq q$,

$$(3) \mathcal{NCT}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m)) \neq \emptyset.$$

Proof. By Theorem 5.1, it suffices to check when

$$M_m = \Psi_q(\hat{\omega}_{p_1, q_1}(u_1) \dots \hat{\omega}_{p_m, q_m}(u_m)) \neq 0.$$

If the three conditions hold, then M_m is of the form given by Lemma 4.2 and since each $b_q(\pi, f)$ contributing to it is a positive number, we get $M_m \neq 0$. Conversely, if $T_{p_i, q_i}(u, n_i)$ is evanescent for some $i \in [m]$, then the corresponding $\hat{\omega}_{p_i, q_i}(u_i)$ is trivial and thus $M_m = 0$. Next, if $T_{p_i, q_i}(u, n_i)$ is unbalanced for some $i \in [m]$, where $q \notin \{p_i, q_i\}$, then, assuming (without loss of generality) that the corresponding symmetrized Gaussian operator is of the form $\hat{\omega}_{p_i, q_i}(u_i) = \omega_{p_i, q_i}(u_i)$ (and thus $d_{q_i} = 0$) and taking the largest such i , we observe that it must act on Ω_{q_i} or on a vector of the form $e_{q_i, t}(u) \otimes w$ in order to give a non-zero result. However, by assumption $q_i \neq q$ and thus $\Omega_{q_i} \neq \Omega_q$ and $e_{q_i, t}(u)$ could only be created by $\omega_{q_i, t}(u)$, which vanishes since $d_{q_i} = 0$. Thus, each unbalanced block has to be of the form $T_{p, q}(u, n)$ for some $p \in [r]$ and $u \in \mathcal{U}$. Finally, if $\mathcal{NCT}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m)) = \emptyset$, then $M_m = 0$ by Lemma 4.2. This completes the proof. \blacksquare

Example 5.4. If $v_{p,q} = v_{q,p} = 1$, then the moments studied in Example 5.2 reduce to polynomials in asymptotic dimensions d_p, d_q . For instance, we obtain

$$\Psi_q(\hat{\omega}_{p,q}^6) = d_p^3 + 3d_p^2d_q + d_pd_q^2$$

by counting blocks of odd and even depths in the following set of partitions:



The moment $\Psi_p(\hat{\omega}_{p,q}^6)$ is obtained from the above by interchanging d_p and d_q . Thus,

$$\Psi(\hat{\omega}_{p,q}^6) = 2d_pd_q^3 + 6d_p^2d_q^2 + 2d_p^3d_q$$

since $\Psi = \sum_j d_j \Psi_j$ and $\Psi_j(\hat{\omega}_{p,q}^6) = 0$ for $j \notin \{p, q\}$. Moreover, the moment under Ψ_q does not vanish even if $d_p > 0$ and $d_q = 0$, which corresponds to the situation in which block $\hat{\omega}_{p,q}$ is unbalanced. Note that the conditions of Corollary 5.3 are satisfied. In particular, the first partition in the picture given above satisfies the third condition and contributes d_p^3 .

In particular, if \mathcal{U} is finite, we can consider distributions of the sums of independent off-diagonal and unbalanced symmetric blocks

$$T_{p,q}(n) = \sum_{u \in \mathcal{U}} T_{p,q}(u, n)$$

under the partial trace $\tau_q(n)$ associated with $d_q = 0$ and $d_p \neq 0$. Under more general assumptions, distributions of such matrices were studied by Benaych-Georges by means of the so-called rectangular convolution [6]. Using Theorem 5.1, we can see that blocks $T_{p,q}(u, n)$ behave asymptotically under $\tau_q(n)$ as operators $\omega_{p,q}(u)$ under the state Ψ_q . This gives random matrix models corresponding to arrays $(\omega_{p,q}(u))$ in which

$$\omega_{p,q}(u) \neq 0 \text{ and } \omega_{q,p}(u) = 0$$

for fixed q and any $p \neq q$ and $u \in \mathcal{U}$. Of course, if $d_q = 0$, then the diagonal symmetric blocks $T_{q,q}(u, n)$ become zero as $n \rightarrow \infty$ for all $u \in \mathcal{U}$ under any partial trace.

Remark 5.2. This fact allows us to give a random matrix model for non-trivial off-diagonal operators $\omega_{p,q}(u)$ in the case of arbitrary dimension matrix. Namely, if $T_{p,q}(u, n)$ consists of a block $S_{p,q}(u, n)$ and of its adjoint, where $S_{p,q}(u, n)$ is expressed in terms of its columns as

$$S_{p,q}(u, n) = (K_1(u, n) \dots K_{n_q}(u, n)),$$

then $\omega_{p,q}(u)$ is the limit under $\tau_q(n)$ as $n \rightarrow \infty$ of the symmetric subblock of $T_{p,q}(u, n)$ which consists of the column $K_j(u, n)$ and of its adjoint for arbitrary j . In the case when all block variances are normalized by n , it does not mean that one can construct a random matrix model for any matrix $(\omega_{p,q}(u))$ since if one entry of this matrix is zero, all entries in the same row must be zero. However, if we consider more general normalizations, this goal can probably be achieved.

6. DOUBLE CONVOLUTION FORMULA

The limit distributions of sums of independent Gaussian square random matrices with complex identically distributed variables are given by the free additive convolution [32,33]. In the case when the variables are block-identically distributed and \mathcal{U} is finite, in order to express the limit distributions of the sums

$$Y(n) = \sum_{u \in \mathcal{U}} Y(u, n)$$

under partial traces in a convolution form, we need to generalize Voiculescu's result and introduce the convolution associated with matricial freeness.

This convolution, called the matricially free convolution, reminds the *strongly matricially free convolution*

$$\boxplus_{p,q} \mu_{p,q} = \text{distribution of } \sum_{p,q} a_{p,q}$$

in the state φ , where $(a_{p,q})$ is an $r \times r$ array of strongly matricially free random variables with respect to an array of states built from a distinguished state φ and its conjugate states, and $(\mu_{p,q})$ is the corresponding array of distributions.

The concept of strong matricial freeness and the associated convolution was studied in detail in [22,24]. We have shown that it unifies the main types of noncommutative independence in the sense that various noncommutative random variables can be decomposed in terms of strongly matricially free ones. One of the consequences of this unification is that both free and boolean convolutions can be expressed in terms of the strongly matricially free convolution of an array of distributions.

In particular, if the array $(a_{p,q})$ is square and of dimension at least two and the corresponding distributions $(\mu_{p,q})$ are row-identical, then

$$\boxplus_{p,q} \mu_{p,q} = \mu_1 \boxplus \mu_2 \boxplus \dots \boxplus \mu_r,$$

where $\mu_{p,q} = \mu_p$ for any p, q , which justifies the notation since in this case the convolution $\boxplus_{p,q}$ can be viewed as a decomposition of the free convolution. However, if the array $(a_{p,q})$ is diagonal, then

$$\boxplus_{p,q} \mu_{p,q} = \mu_1 \uplus \mu_2 \uplus \dots \uplus \mu_r,$$

where $\mu_{p,q} = \mu_p$ whenever $p = q$. It should be remarked that the strongly matricially free convolution is not just the free convolution taken over an array of indices. These

properties of the strongly matricially free convolution will be used in this section. In the meantime, let us introduce another convolution of matricial type.

Definition 6.1. Let $(a_{p,q})$ be an array of variables which are matricially free with respect to the array $(\varphi_{p,q})$, where $\varphi_{p,q} = \varphi_q$ for any $(p,q) \in \mathcal{J}$, with distributions $(\mu_{p,q})$, respectively. By the *matricially free convolution* of $(\mu_{p,q})$ we shall understand

$$\boxtimes_{p,q} \mu_{p,q} = \text{distribution of } \sum_{p,q} a_{p,q}$$

in the state $\psi = \sum_q d_q \varphi_q$, where $d_q \geq 0$ for any q , where we suppress its dependence on ψ or on the dimension matrix D .

The most important examples of matricially free convolutions are the distributions of ‘independent’ *Gaussian pseudomatrices* [23]

$$\omega(u) = \sum_{p,q} \omega_{p,q}(u)$$

which generalize free Gaussian operators, and the distributions of their sums

$$\omega = \sum_{u \in \mathcal{U}} \omega(u)$$

in the state Ψ , where the associated arrays of distributions are matricial semicircle laws.

Example 6.1. By [23, Proposition 4.2], where we have just one matrix of matricially free Gaussian operators, the distribution of $\omega = \sum_{p,q} \omega_{p,q}$ in the state $\Psi = \sum_q d_q \Psi_q$ is given by $\mu = \boxtimes_{p,q} \mu_{p,q}$, where

$$\mu_{p,q} = \begin{cases} \sigma_{q,q} & \text{if } p = q \\ \kappa_{p,q} & \text{if } p \neq q \end{cases},$$

which corresponds to the simplest case of Proposition 2.3, when the set \mathcal{U} consists of one element.

In order to express the distributions of the sum of matricially free Gaussian operators in a convolution form in the general case, we shall introduce a new notation. If $(\mu_{p,q})$ and $(\nu_{p,q})$ are arrays of distributions indexed by \mathcal{J} , we shall denote

$$\mu_{p,q} \boxplus_{\mathcal{J}} \nu_{p,q} = \begin{cases} \mu_{q,q} \boxplus \nu_{q,q} & \text{if } p = q \\ \mu_{p,q} \oplus \nu_{p,q} & \text{if } p \neq q \end{cases}$$

where $\mu \boxplus \nu$ and $\mu \oplus \nu$ denote the free convolution and the boolean convolution, respectively. It can be seen that this convolution is pointwise associative, i.e. associative for fixed $(p,q) \in \mathcal{J}$ since both free and boolean convolutions are associative.

Lemma 6.1. *Under the assumptions of Proposition 2.3, the Ψ -distribution of the sum of collective Gaussian operators $\omega = \sum_{p,q} \omega_{p,q}$ is given by*

$$\mu = \boxtimes_{p,q} (\mu_{p,q}(1) \boxplus_{\mathcal{J}} \dots \boxplus_{\mathcal{J}} \mu_{p,q}(m))$$

where the arrays $(\mu_{p,q}(u))$, $u \in \mathcal{U} = [m]$, are given by Corollary 2.1.

Proof. We will first prove that the family

$$\{\omega_{p,q}(1), \dots, \omega_{p,q}(m)\}$$

is free (boolean independent) with respect to $\varphi_{p,q} = \Psi_q$ for any fixed diagonal (off-diagonal) $(p, q) \in \mathcal{J}$. Let $\mathcal{N}_{p,q}(u)$ be the subspace of $\mathcal{N}_{p,q}$ spanned by tensors which begin with $e_{p,q}(u)$ and let $s_{p,q}(u)$ be the associated orthogonal projection. First, we will treat the off-diagonal case. Denote, for simplicity, $\wp(u) = \wp_{p,q}(u)$, $\wp^*(u) = \wp_{p,q}^*(u)$, $s(u) = s_{p,q}(u)$ and $1(u) = 1_{p,q}(u)$, where $1_{p,q}(u)$ is the orthogonal projection onto

$$\mathbb{C}\Omega_q \oplus \mathcal{N}_{p,q}(u) \oplus \bigoplus_k \mathcal{N}_{q,k},$$

thus note that $1_{p,q}(u)$ is slightly smaller than $1_{p,q}$ of Definition 2.5, and let

$$\mathcal{A}(u) := \text{alg}(\wp(u), \wp^*(u), 1(u))$$

for any $u \in \mathcal{U}$. We will show that

$$\Psi_q(a_1 \dots a_n) = \Psi_q(a_1) \dots \Psi_q(a_n)$$

for any $a_k \in \mathcal{A}(u_k)$, where $u_1, \dots, u_n \in \mathcal{U}$. First, let us check how a_n acts onto Ω_q . Since a_n is a linear combination of $\wp(u_n)$, $\wp^*(u_n)$, $s(u_n)$ and $1(u_n)$, we clearly have

$$a_n \Omega_q = \Psi_q(a_n) \Omega_q + \gamma e_{p,q}(u_n)$$

for some $\gamma \in \mathbb{C}$. However, if $l \neq k$, each element of $\mathcal{A}(u_l)$ kills all vectors which begin with $e_{p,q}(u_k)$ by Proposition 2.1. Therefore,

$$\langle a_1 \dots a_n \Omega_q, \Omega_q \rangle = \Psi_q(a_1) \dots \Psi_q(a_n)$$

which shows boolean independence of subalgebras $\mathcal{A}(1), \dots, \mathcal{A}(m)$ with respect to Ψ_q . Therefore, the associated Gaussian operators are also boolean independent. Now, let us consider the diagonal case and fix $(q, q) \in \mathcal{J}$. Again, denote $\wp(u) = \wp_{q,q}(u)$ and $\wp^*(u) = \wp_{q,q}^*(u)$ for any u . In this case, we take the same unit $1_{q,q}$ for each u , and we denote

$$\mathcal{A}(u) = \text{alg}(\wp(u), \wp^*(u))$$

where we remark that we do not need to put the unit $1_{q,q}$ among the generators since $\wp_{q,q}^*(u) \wp_{q,q}(u) = b_{q,q}(u) 1_{q,q}$ and we assume that $b_{q,q}(u) > 0$. Using the relations of Proposition 2.2, we can write every element of $\mathcal{A}(u) \cap \text{Ker} \Psi_q$ in the form of a linear combination of

$$\wp^p(u) \wp^{*q}(u), \text{ where } p + q > 0.$$

However, since $\wp^*(u)$ kills Ω_q and, moreover, $\wp^*(u) \wp(t) = 0$ for any $u \neq t$, in order to prove the freeness condition

$$\Psi_q(a_1 \dots a_n) = 0$$

for any $a_k \in \mathcal{A}(u_k) \cap \text{Ker} \Psi_q$, it suffices to observe that

$$\Psi_q(\wp^{p_1}(u_1) \dots \wp^{p_m}(u_m)) = 0.$$

Finally, matricial freeness of the array of algebras $\mathcal{A}_{p,q} = \text{alg}(\wp_{p,q}, \wp_{p,q}^*, 1_{p,q})$ with respect to $(\varphi_{p,q})$ follows from Proposition 2.3, and that entails matricial freeness of the array $(\omega_{p,q})$. This completes the proof. \blacksquare

Definition 6.2. Let $(a_{i,j})$ be a symmetric array of variables which are symmetrically matricially free with respect to the array $(\varphi_{i,j})$, where $\varphi_{i,j} = \varphi_j$ for any i, j , with

distributions $(\mu_{i,j})$, respectively. By the *symmetrically matricially free convolution* of $(\mu_{i,j})$ we will understand

$$\widehat{\boxtimes}_{i,j} \mu_{i,j} = \text{distribution of } \sum_{i \leq j} a_{i,j}$$

in the state $\psi = \sum_j d_j \varphi_j$, where we suppress its dependence on the dimension matrix D .

Lemma 6.2. *Under the assumptions of Proposition 3.5, the Ψ -distribution of the sum $\widehat{\omega} = \sum_{u=1}^m \widehat{\omega}(u)$ is given by*

$$\mu = \widehat{\boxtimes}_{p,q}(\sigma_{p,q}(1) \boxplus \dots \boxplus \sigma_{p,q}(m))$$

where $(\sigma_{p,q}(u))$ is the array of semicircle laws of Corollary 3.1 for any $u \in \mathcal{U}$.

Proof. By Proposition 3.5, the array of variables

$$\widehat{\omega}_{p,q} = \sum_{u=1}^m \widehat{\omega}_{p,q}(u)$$

is symmetrically matricially free with respect to the array $(\Psi_{p,q})$ defined by Ψ_1, \dots, Ψ_r . Moreover, the distribution of $\widehat{\omega}_{p,q}(u)$ in the state $\Psi_{p,q}$ is the semicircle law of radius $2\alpha_{p,q}(u)$ for any $(p,q) \in \mathcal{J}$ and $u \in \mathcal{U}$. Therefore, it suffices to show that for any given $(p,q) \in \mathcal{J}$, the family of variables

$$\{\widehat{\omega}_{p,q}(1), \dots, \widehat{\omega}_{p,q}(m)\}$$

is free with respect to Ψ_q . The proof for the case when $p = q$ is the same as in Lemma 6.1 since $\widehat{\omega}_{q,q}(u) = \omega_{q,q}(u)$ for any u . Let us fix $(p,q) \in \mathcal{J}$ and denote $\widehat{\wp}(u) = \widehat{\wp}_{p,q}(u)$ and $\widehat{\wp}^*(u) = \widehat{\wp}_{p,q}^*(u)$ for any u . Let

$$\widehat{\mathcal{A}}(u) = \text{alg}(\widehat{\wp}(u), \widehat{\wp}^*(u))$$

where, as in the diagonal case, we do not need to include the unit $1_{p,q}$ among the generators since $\widehat{\wp}_{p,q}^*(u) \widehat{\wp}_{p,q}(u) = b_{p,q}(u) 1_{p,q}$ and $b_{p,q}(u) > 0$. Using the commutation relations of Proposition 3.4, we can write every element of $\widehat{\mathcal{A}}(u) \cap \text{Ker} \Psi_q$ in the form of a linear combination of

$$\widehat{\wp}^i(u) \widehat{\wp}^{*j}(u), \text{ where } i + j > 0.$$

However, since $\widehat{\wp}^*(u)$ kills Ω_q and, moreover, $\widehat{\wp}^*(u) \widehat{\wp}(t) = 0$ for any $u \neq t$, in order to prove the freeness condition

$$\Psi_q(a_1 \dots a_n) = 0$$

for any $a_k \in \widehat{\mathcal{A}}(u_k) \cap \text{Ker} \Psi_q$, it suffices to observe that

$$\Psi_q(\widehat{\wp}^{p_1}(u_1) \dots \widehat{\wp}^{p_m}(u_m)) = 0,$$

which completes the proof. ■

We choose Lemma 6.1 to find a convolution formula for the limit law of the sum of independent Hermitian Gaussian random matrices. In the identically distributed case, we obtain the free convolution of semicircle distributions [32]. In our generalization of that result, the free convolution is replaced by the strongly matricially free convolution. In addition, we obtain the matricially free convolution which tells us how to convolve asymptotic distributions of different blocks. These distributions are computed with

respect to partial traces or with respect to their convex linear combination given by the trace. We can call this formula the *Double Convolution Formula* since it involves both the matricially free convolution and the strongly matricially free one.

Equivalently, one can work with the symmetrically matricially free convolution. In particular, one can use Lemma 6.2 to prove a similar convolution formula to that of Theorem 6.1. However, an analytic approach to the symmetrically matricially free convolution is more complicated and will not be treated in this paper. For that reason, we restrict ourselves to the case of the matricially free convolution in the theorem given below as well as in Section 8, where we discuss some analytic results.

We shall use the square roots of all entries of $B(u)$, namely

$$\alpha_{p,q}(u) = \sqrt{b_{p,q}(u)}$$

for any $(p, q) \in \mathcal{J}$ and $u \in \mathcal{U}$, with the corresponding matrices denoted $A(u) = (\alpha_{p,q}(u))$. Moreover, in order to express the result in terms of the strongly matricially free convolution, we shall use pairs $(r, s) \in \mathcal{U} \times \mathcal{U}$ instead of $u \in \mathcal{U}$, and for that purpose we set $\alpha_{q,q}(r, s) = \alpha_{q,q}(r)$ and $\alpha_{p,q}(r, r) = \alpha_{p,q}(r)$ for any $p \neq q$ and any $r, s \in \mathcal{U}$.

Theorem 6.1. *Under the assumptions of Theorem 5.1, the limit law of the sum of independent Hermitian Gaussian random matrices*

$$Y(n) = \sum_{u \in \mathcal{U}} Y(u, n)$$

under the trace takes the convolution form

$$\mu = \boxplus_j \boxplus_{\mathcal{K}} \mu_j(\mathcal{K})$$

where $j \in \mathcal{J}$ and $\mathcal{K} \in \mathcal{U} \times \mathcal{U}$ and

- (1) $\mu_j(\mathcal{K}) = \sigma_j(\mathcal{K})$ for any $\mathcal{K} \in \mathcal{U} \times \mathcal{U}$ whenever $j \in \mathcal{J}$ is diagonal
- (2) $\mu_j(\mathcal{K}) = \kappa_j(\mathcal{K})$ for any diagonal $\mathcal{K} \in \mathcal{U} \times \mathcal{U}$ whenever $j \in \mathcal{J}$ is off-diagonal,

where $\sigma_j(\mathcal{K})$ is the semicircle law of radius $2\alpha_j(\mathcal{K})$ and $\kappa_j(\mathcal{K})$ is the Bernoulli law concentrated at $\pm\alpha_j(\mathcal{K})$.

Proof. We have

$$Y^m(n) = \sum_{u_1, \dots, u_m \in \mathcal{U}} \sum_{p_1, q_1, \dots, p_m, q_m} T_{p_1, q_1}(u_1, n) \dots T_{p_m, q_m}(u_m, n)$$

and thus, using Theorem 5.1, we obtain

$$\lim_{n \rightarrow \infty} \tau(n)(Y^m(n)) = \Psi(\omega^m)$$

for any $m \in \mathbb{N}$, where

$$\omega = \sum_{p,q} \omega_{p,q} = \sum_{p,q} \sum_{u \in \mathcal{U}} \omega_{p,q}(u).$$

Now, by Lemma 6.1, the Ψ -distribution of ω is given by

$$\boxplus_{p,q} (\mu_{p,q}(1) \boxplus_j \dots \boxplus_j \mu_{p,q}(m)),$$

where $(\mu_{p,q}(u))$ is the array of semicircle and Bernoulli laws of Corollary 2.1 for any $u \in \mathcal{U}$ and any p, q . It remains to decompose the free and boolean convolutions in terms of the strongly matricially free convolutions [24, Theorem 3.1]. Using the above

notation, where the pair (p, q) is already reserved for the matricially free convolution, we can write these decompositions in the form

$$\begin{aligned}\mu_{q,q}(1) \boxplus \dots \boxplus \mu_{q,q}(m) &= \boxplus_{r,s} \gamma_{q,q}(r, s) \\ \mu_{p,q}(1) \uplus \dots \uplus \mu_{p,q}(m) &= \boxplus_{r,s} \gamma_{p,q}(r, s),\end{aligned}$$

where $\gamma_{q,q}(r, s) = \mu_{q,q}(r)$ for any $1 \leq r, s \leq m$ and any $(q, q) \in \mathcal{J}$ and $\gamma_{p,q}(r, r) = \mu_{p,q}(r)$ for any $1 \leq r \leq m$ and $\gamma_{p,q}(r, s) = \delta_0$ when $r \neq s$, for any off-diagonal $(p, q) \in \mathcal{J}$. This completes the proof. \blacksquare

Corollary 6.1. *Under the assumptions of Theorem 6.1,*

(1) *if $r = 1$, then*

$$\mu = \boxplus_{\kappa} \mu(\kappa),$$

which corresponds to the free convolution of semicircle laws.

(2) *if $m = 1$, then*

$$\mu = \boxplus_j \mu_j,$$

which corresponds to the matricially free convolution of semicircle and Bernoulli laws which form one matricial semicircle law.

Proof. The first statement holds since in the case when $r = 1$, all matrices have one diagonal symmetric block whose complex entries are identically distributed and for $r = 1$ the strongly matricially free convolution over κ reduces to the free convolution. The second statement restates the result of [23] on the distribution of one Gaussian random pseudomatrix $\omega = \sum_{p,q} \omega_{p,q}$ in the state Ψ . \blacksquare

In the general case, when both convolutions are present, we can interpret their origins as follows:

- (a) convolution \boxplus_{κ} describes the asymptotic relation between symmetric blocks having the same position in independent matrices,
- (b) convolution \boxplus_j describes the asymptotic relation between symmetric blocks having different positions in a given matrix,

where by an independent matrix we understand an independent Hermitian Gaussian random matrix.

Remark 6.1. It should be noted that the limit distribution of $Y(n)$ under the trace $\tau(n)$ given by the Double Convolution Formula depends on the dimension matrix D . In fact, D uniquely determines the convolution \boxplus . However, it is not hard to see that a similar formula holds also for limit distributions under the partial traces $\tau_q(n)$, where $q \in [r]$. In fact, if $d_q = 0$, the moments under Ψ_q do not contribute to the asymptotic distributions under the trace whereas they may be non-trivial as Example 5.4 demonstrates (see also Section 8). To avoid ambiguity, we prefer to speak of the convolution \boxplus under the state Ψ or under the state Ψ_q , where $q \in [r]$, instead of describing \boxplus in terms of D .

Remark 6.2. Both boolean and free convolutions are associative, and thus we can define an associative convolution of arrays $[\mu] = (\mu_{p,q})$, $[\nu] = (\nu_{p,q})$, where $(p, q) \in \mathcal{J}$, of probability measures on the real line

$$[\mu] \boxplus [\nu] = [\gamma]$$

where

$$\gamma_{p,q} = \mu_{p,q} \boxplus_{\mathcal{J}} \nu_{p,q}$$

for any $(p, q) \in \mathcal{J}$, where we use the notation \boxplus as a generalization of the free convolution to the family of arrays. Using this convolution, one can rephrase the Double Convolution Formula as follows. The sum

$$Y(n) = \sum_{u=1}^m Y(n, u)$$

of independent Hermitian Gaussian random matrices converges in moments to the pseudomatrix ω , meaning that the moments of $Y(n)$ under $\tau_q(n)$ converge to the moments of ω under Ψ_q for any $q \in [r]$, and thus converge to the corresponding moments of the free convolution of matricial semicircle laws

$$[\sigma] = [\sigma(1)] \boxplus [\sigma(2)] \boxplus \dots \boxplus [\sigma(m)].$$

This gives convergence in moments for any convex linear combination of the considered states. One can also define the associated notion of independence for arrays of noncommutative probability spaces, but we will not discuss it here.

7. ASYMPTOTIC MONOTONE INDEPENDENCE AND S-FREENESS

The Double Convolution Formula seems to suggest that other convolutions decomposable in terms of the strongly matricially free convolution might be obtained as limit distributions of random matrix models using the framework of matricial freeness. In the case of block-identically distributed arrays, we were able to obtain only free and boolean convolutions as the limit distributions of the diagonal and off-diagonal entries, respectively, of the sum of Hermitian Gaussian random matrices.

In this section we give two other examples, the monotone and s-free convolutions. However, in order to do that, one has to recall that they arise in the context of the decompositions of the free convolution in terms of other convolutions [21] and that they can be obtained from the strongly matricially free convolution [22]. Therefore, using these decompositions, it is natural to decompose independent Hermitian Gaussian matrices with identically distributed entries in terms of symmetric blocks and then suitably identify the right pieces of these decompositions with random variables independent in the required sense.

We will restrict our attention to the case of two independent Hermitian Gaussian random matrices decomposed in terms of their three symmetric blocks as

$$Y(u, n) = \sum_{1 \leq i \leq j \leq 2} T_{i,j}(u, n),$$

where $u \in \{1, 2\}$. Choosing appropriate symmetric blocks of these matrices, we will construct pairs of independent Hermitian Gaussian random matrices $\{X(1, n), X(2, n)\}$ which are asymptotically boolean independent, monotone independent or s-free under one of the partial traces. Thus, the distributions of the sums

$$X(n) = X(1, n) + X(2, n)$$

will be asymptotically boolean, monotone or s-free convolutions of the asymptotic distributions of the summands. The boolean case follows immediately from Lemma 6.1, but the other two are slightly more involved.

In view of Theorem 5.1, if we are able to construct families of random variables which are independent in some sense from a family of arrays of matricially free Gaussian operators, a corresponding random matrix model is close at hand. The definitions of boolean independence, monotone independence and s-freeness can be found in [22]. Let us only remark that in the last two cases the order in which the variables appear is relevant.

Proposition 7.1. *If $\mathcal{U} = \{1, 2\}$, $r = 2$ and $b_{p,q}(u) = 1$ for any $p, q, u \in [2]$, then*

- (1) *the pair $\{\omega_{1,2}(1), \omega_{1,2}(2)\}$ is boolean independent with respect to Ψ_2 ,*
- (2) *the pair $\{\omega_{1,2}(1), \omega_{1,2}(2) + \omega_{1,1}(2)\}$ is monotone independent with respect to Ψ_2 ,*
- (3) *the pair $\{\omega_{1,2}(1) + \omega_{1,1}(1), \omega_{1,1}(2)\}$ is s-free with respect to (Ψ_2, Ψ_1) .*

Proof. (1) follows from Lemma 6.1. To prove (2), denote

$$a = \omega_{1,2}(1), \quad b = \omega_{1,2}(2) + \omega_{1,1}(2).$$

We need to show that the pair $\{a, b\}$ (in that order) is monotone independent w.r.t. Ψ_2 , i.e.

$$\Psi_2(w_1 a_1 b_1 a_2 w_2) = \Psi_2(b_1) \Psi_2(w_1 a_1 a_2 w_2)$$

for any $a_1, a_2 \in \mathbb{C}[a, 1_1]$, $b_1 \in \mathbb{C}[b, 1_2]$, where $1_1 = 1_{1,2}$ and $1_2 = 1$ and w_1, w_2 are arbitrary elements of $\mathbb{C}\langle a, b, 1_1, 1_2 \rangle$. It suffices to consider the action of a and b onto their invariant subspace in \mathcal{M} of the form

$$\mathcal{M}' = \mathbb{C}\Omega_2 \oplus (\mathcal{F}(2) \otimes \mathcal{H}(1)) \oplus (\mathcal{F}(2) \otimes \mathcal{H}(2))$$

where $\mathcal{F}(2) = \mathcal{F}(\mathbb{C}e_{1,1}(2))$ with the vacuum vector Ω and $\mathcal{H}(u) = \mathbb{C}e_{1,2}(u)$ for $u \in \{1, 2\}$, where we identify $\Omega \otimes e_{1,2}(u)$ with $e_{1,2}(u)$. Now, the range of any polynomial in a is $\mathbb{C}\Omega_2 \oplus \mathcal{H}(1)$ since

$$a^k \Omega_2 = \begin{cases} \Omega_2 & \text{if } k \text{ even} \\ e_{1,2}(1) & \text{if } k \text{ odd} \end{cases}$$

and $1_1 \Omega_2 = \Omega_2$, $1_1 e_{1,2}(1) = e_{1,2}(1)$. Therefore, it suffices to compute the action of any polynomial in b onto Ω_2 and $e_{1,2}(1)$. Now, the action of powers of b onto Ω_2 and onto $e_{1,2}(1)$ is the same as the action of the free Gaussian operator onto the vacuum vector in the free Fock space. Namely, we have

$$b^{2k} \Omega_2 = C_k \Omega_2 \bmod (\mathcal{M}' \ominus \mathbb{C}\Omega_2)$$

and

$$b^{2k} e_{1,2}(1) = C_k e_{1,2}(1) \bmod (\mathcal{M}' \ominus (\mathbb{C}\Omega_2 \oplus \mathcal{H}(1)))$$

where C_k is the k th Catalan number and

$$b^{2k-1} \Omega_2 = b^{2k-1} e_{1,2}(1) = 0 \bmod (\mathcal{M}' \ominus (\mathbb{C}\Omega_2 \oplus \mathcal{H}(1)))$$

for any $k \in \mathbb{N}$. Thus $\Psi_2(b^{2k}) = C_k$ and, moreover, since $\mathcal{M}' \ominus (\mathbb{C}\Omega_2 \oplus \mathcal{H}(1)) \subset \text{Ker } a$, the required condition for monotone independence holds if b_1 is a positive power of b . It is easy to see that it also holds if $b_1 = 1_2$, which completes the proof of (2). To prove (3), consider

$$a = \omega_{1,2}(1) + \omega_{1,1}(1), \quad b = \omega_{1,1}(2)$$

and their invariant subspace in \mathcal{M} of the form

$$\mathcal{M}'' = \mathbb{C}\Omega_2 \oplus (\mathcal{F}(1 \oplus 2) \otimes \mathcal{H}(1))$$

where $\mathcal{F}(1 \oplus 2) = \mathcal{F}(\mathbb{C}e_{1,1}(1) \oplus \mathbb{C}e_{1,1}(2))$. Observe that \mathcal{M}'' is isomorphic to

$$\mathcal{F}_{1,2} = \mathbb{C}\Omega \oplus \mathcal{F}_1^0 \oplus (\mathcal{F}_2^0 \otimes \mathcal{F}_1^0) \oplus (\mathcal{F}_1^0 \otimes \mathcal{F}_2^0 \otimes \mathcal{F}_1^0) \oplus \dots$$

where $\mathcal{F}_j^0 = \mathcal{F}(\mathbb{C}e_j) \ominus \mathbb{C}\xi_j$ and the isomorphism τ is given by $\tau(\Omega_2) = \Omega$ and

$$\tau(e_{1,1}(j_1) \otimes \dots \otimes e_{1,1}(j_{k-1}) \otimes e_{1,2}(1)) = e_{j_1} \otimes \dots \otimes e_{j_{k-1}} \otimes e_1$$

for any $j_1, \dots, j_{k-1} \in \{1, 2\}$ and $k \in \mathbb{N}$. The space $\mathcal{F}_{1,2}$ is the s-free product of two free Fock spaces, (\mathcal{F}_1, ξ_1) and (\mathcal{F}_2, ξ_2) , the subspace of their free product $(\mathcal{F}_1, \xi_1) * (\mathcal{F}_2, \xi_2)$ (for the definition of the s-free product of Hilbert spaces and of the s-free convolution describing the free subordination property, see [21]). Now, observe that the action of a onto \mathcal{M}'' can be identified with the action of the canonical free Gaussian $\omega(e_1)$ restricted to $\mathcal{F}_{1,2}$. Similarly, the action of b onto \mathcal{M}'' can be identified with the action of $\omega(e_2)$ restricted to $\mathcal{F}_{1,2}$. This proves that the pair $\{a, b\}$ (in that order) is s-free with respect to the pair of states (Ψ_2, Ψ_1) , which gives (3). \blacksquare

We are ready to state certain results concerning asymptotic properties of independent Gaussian random matrices $X(1, n), X(2, n)$ built from at most three symmetric blocks of $Y(1, n), Y(2, n)$, respectively. We assume that one asymptotic dimension is equal to zero, thus one block is balanced, one is unbalanced and one is evanescent.

Theorem 7.1. *Under the assumptions of Theorem 5.1, let $\mathcal{U} = \{1, 2\}$, $r = 2$, $d_2 = 0$ and $d_1 = 1$ and let all block variances to be equal to one. Then*

- (1) *the pair $\{T_{1,2}(1, n), T_{1,2}(2, n)\}$ is asymptotically boolean independent with respect to $\tau_2(n)$,*
- (2) *the pair $\{T_{1,2}(1, n), Y(2, n)\}$ is asymptotically monotone independent with respect to $\tau_2(n)$,*
- (3) *the pair $\{Y(1, n), T_{1,1}(2, n)\}$ is asymptotically s-free with respect to $(\tau_2(n), \tau_1(n))$.*

Proof. Since $d_1 = 1$ and $d_2 = 0$, blocks $T_{1,1}(u, n), T_{1,2}(u, n)$ and $T_{2,2}(u, n)$ are balanced, unbalanced and evanescent, respectively. Therefore, by Theorem 5.1, we have convergence as $n \rightarrow \infty$

$$\begin{aligned} T_{1,2}(u, n) &\rightarrow \omega_{1,2}(u) \\ T_{1,1}(u, n) &\rightarrow \omega_{1,1}(u) \\ T_{2,2}(u, n) &\rightarrow 0 \end{aligned}$$

in the sense of moments under $\tau_2(n)$ for $u \in \{1, 2\}$, where the limit moments are computed in the state Ψ_2 . By Proposition 7.1, the proof is completed. \blacksquare

8. CATALAN MATRICES

In general, finding explicit analytic formulas for the limit distributions given by the Double Convolution Formula is difficult, if at all possible. However, one can prove existence of a matrix-valued reciprocal Cauchy transform whose coefficients are matricial analogs of the Catalan numbers. Moreover, certain formulas for the scalar-valued Cauchy transforms of these distributions or their reciprocals can also be derived.

Let us recall that the convolutions of Theorem 6.1 are distributions of the operator of the form

$$\omega = \sum_{p,q} \omega_{p,q}$$

in the state

$$\Psi = \sum_{q=1}^r d_q \Psi_q$$

where each $\omega_{p,q}$ is a collective Gaussian operator defined as in the proof of Theorem 6.1. In view of Lemma 6.1, it is easy to determine the distributions of operators $\omega_{p,q}$ in the states $\Psi_{p,q} = \Psi_q$, respectively, and thus the distribution in the state Ψ .

Proposition 8.1. *Each diagonal collective Gaussian operator $\omega_{q,q}$ has the semicircle distribution of radius*

$$2\alpha_{q,q} = 2 \left(\sum_{u \in \mathcal{U}} \alpha_{q,q}^2(u) \right)^{1/2}$$

in the state Ψ_q for any $q \in [r]$. Each off-diagonal collective Gaussian operator $\omega_{p,q}$ has the Bernoulli distribution concentrated at

$$\pm \alpha_{p,q} = \pm \left(\sum_{u \in \mathcal{U}} \alpha_{p,q}^2(u) \right)^{1/2}$$

in the state Ψ_q for any $p \neq q$.

Proof. Each diagonal operator $\omega_{q,q}(u)$ has the semicircle distribution of radius $2\alpha_{q,q}(u)$ for any $q \in [r]$ and $u \in \mathcal{U}$ by Proposition 2.3. The family $\{\omega_{q,q}(u), u \in \mathcal{U}\}$ is free w.r.t. Ψ_q for any q by Lemma 6.1. Therefore, the corresponding sum $\omega_{q,q}$ has the semicircle distribution of radius $2\alpha_{q,q}$ given above. In turn, since each off-diagonal operator $\omega_{p,q}(u)$ has the Bernoulli distribution concentrated at $\pm \alpha_{p,q}(u)$ and the family $\{\omega_{p,q}(u), u \in \mathcal{U}\}$ is boolean independent w.r.t. Ψ_q for any $p \neq q$, the corresponding sum $\omega_{p,q}$ has the Bernoulli distribution concentrated at $\pm \alpha_{p,q}$ given above. ■

Therefore, the array $(\omega_{p,q})$ of collective Gaussian operators is of the same type as in the case of one array. This implies that the results on the Cauchy transforms of the limit distributions of random pseudomatrices (sums of matricially free random variables) studied in [22] can be used in our random matrix context. In fact, we showed in [23] that symmetric blocks of a Hermitian Gaussian random matrix of Theorem 5.1 have the same limit joint distributions as blocks of a random pseudomatrix. Therefore, the distributions given by the continued fractions of Theorem 8.1 coincide with the matricially free convolutions of the distributions of $(\omega_{p,q})$.

Theorem 8.1. *The Cauchy transform of the distribution of the operator ω in the state Ψ_q , where $q \in [r]$, takes the form*

$$G_{\mu_q}(z) = \frac{1}{z - \sum_p \frac{b_{p,q}}{z - \sum_k \frac{b_{k,p}}{z - \sum_j \frac{b_{j,k}}{z - \dots}}}}$$

where $b_{p,q} = \sum_{u \in \mathcal{U}} \alpha_{p,q}^2(u)$ for any $p, q \in [r]$ and where the continued fraction converges uniformly on the compact subsets of \mathbb{C}^+ to a probability measure μ_q on the real line with compact support.

Proof. We have shown in [23, Theorem 7.1] that mixed moments of blocks of pseudomatrices, converge under partial traces $\psi_q(n)$ to the mixed moments of the matricially free Gaussian operators under Ψ_q , respectively. These operators have semicircle or Bernoulli distributions and are matricially free by [23, Proposition 4.2], or by the generalized version of this result given in Proposition 2.3 of this paper. Therefore, since the collective Gaussian operators $\omega_{p,q}$ have semicircle or Bernoulli distributions by Proposition 8.1 and the array $(\omega_{p,q})$ is matricially free with respect to $(\Psi_{p,q})$ by Proposition 2.3, we can treat them as the limit operatorial realizations of blocks of certain random pseudomatrices under partial traces. We showed in [22, Theorem 7.1] that the distributions of such blocks under partial traces converge weakly to the measures μ_1, \dots, μ_r on the real line whose Cauchy transforms are of the form given above. This proves our assertion. \blacksquare

Corollary 8.1. *The measures of Theorem 8.1 take the boolean convolution form*

$$\mu_q = \mu_{1,q} \uplus \mu_{2,q} \uplus \dots \uplus \mu_{r,q}$$

for any $q \in [r]$, where $(\mu_{i,j})$ is the array of probability measures on the real line whose K -transforms are defined by continued fractions

$$K_{p,q}(z) = \frac{b_{p,q}}{z - \sum_k \frac{b_{k,p}}{z - \sum_j \frac{b_{j,k}}{z - \dots}}}$$

which converge on the compact subsets of \mathbb{C}^+ , where $K_\mu(z) = z - F_\mu(z)$ and $F_\mu(z)$ is the reciprocal Cauchy transform of μ .

Proof. This is an immediate consequence of Theorem 8.1 and [22, Lemma 7.1]. \blacksquare

Finding an analytical formula for G_{μ_q} in the general case does not seem possible. Let us present two situations discussed in [22] in which certain interesting convolution formulas can be derived. From these formulas one can easily obtain the transforms.

Corollary 8.2. *If $A = (\alpha_{p,q})$ is a square r -dimensional matrix with identical positive entries $\alpha_{p,q} = \sqrt{\sum_{u \in \mathcal{U}} \alpha_{p,q}^2(u)} = \alpha_p$ in the p -th row, then*

$$\mu_q = \sigma_{\alpha_1} \boxplus \sigma_{\alpha_2} \boxplus \dots \boxplus \sigma_{\alpha_r}$$

for each $q \in [r]$, where σ_{α_p} is the semicircle distribution of radius $2\alpha_p$ for $p \in [r]$.

Proof. This is an immediate consequence of Theorem 8.1 and [22, Proposition 8.2]. \blacksquare

Example 8.1. Using Corollary 8.2, we can give a simple example of the matricially free convolution. Let us consider the case of one array with all variances equal to one. We convolve the array $(\mu_{p,q})$, where $\mu_{q,q}$ is the semicircle law of radius $2\sqrt{d_q}$ and $\mu_{p,q}$ is

the Bernoulli law concentrated at $\pm\sqrt{d_p}$ for any $p \neq q$. The matricially free convolution of Theorem 6.1 takes the form

$$\mu = \sum_{q=1}^r d_q \mu_q = \sigma_1$$

since $\mu_q = \sigma_1$ for any $q \in [r]$ by Corollary 8.2. In fact, the free convolution of semicircle laws with radii $2\sqrt{d_1}, \dots, 2\sqrt{d_r}$ is the semicircle law with radius one, where we use $d_1 + \dots + d_r = 1$. A similar result can be obtained if we convolve a finite family of arrays.

In the case of two-dimensional square arrays, we can express the distributions μ_1, μ_2 in terms of *s-free additive convolutions*. These convolutions were introduced in [21] in the context of the subordination property of the free additive convolution of probability measures on the real line $\mathcal{M}_{\mathbb{R}}$ discovered by Voiculescu [34] and generalized by Biane [8]. For details on the s-free convolution and the convolution formulation of the subordination property given below, see [22].

Theorem 8.2. *For any $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ there exists a unique $\nu \boxplus \mu \in \mathcal{M}_{\mathbb{R}}$ such that*

$$\mu \boxplus \nu = \mu \triangleright (\nu \boxplus \mu)$$

called the s-free convolution of μ and ν . Moreover, it holds that

$$\mu \boxplus \nu = (\mu \boxplus \nu) \uplus (\nu \boxplus \mu),$$

where $\mu \triangleright \nu$ is the monotone convolution of μ and ν .

Remark 8.1. This is a convolution formulation of the subordination property of the free additive convolution, usually written in terms of the reciprocal Cauchy transforms of probability measures on the real line as

$$F_{\mu \boxplus \nu}(z) = F_{\mu}(F_1(z)) = F_{\nu}(F_2(z))$$

where $F_1(z)$ and $F_2(z)$ are the reciprocal Cauchy transforms of the unique probability measures on the real line which satisfy the above equations. In our convolution formulation, these measures are denoted by $\nu \boxplus \mu$ and $\mu \boxplus \nu$, respectively.

In order to write convolution formulas for the distributions of ω under Ψ_1 and Ψ_2 , we shall use boolean compressions of semicircle laws. For any probability measure μ on the real line, we denote by $T_t \mu$ its boolean compression defined in terms of K-transforms as

$$K_{T_t \mu} = t K_{\mu},$$

where $t \geq 0$. We allow $t = 0$, in which case $T_0 \mu = \delta_0$. We shall use two-parameter boolean compressions of semicircle distributions, $\varsigma_{\alpha, \beta} = T_t \sigma_{\alpha}$ for $t = (\beta/\alpha)^2$, where

$$G_{\varsigma_{\alpha, \beta}}(z) = \frac{(2\alpha^2 - \beta^2)z - \beta^2 \sqrt{z^2 - 4\alpha^2}}{(2\alpha^2 - 2\beta^2)z^2 + 2\beta^4}$$

are their Cauchy transforms, where the branch of the square root is chosen so that $\sqrt{z^2 - 4\alpha^2} > 0$ if $z \in \mathbb{R}$ and $z \in (2\alpha, \infty)$.

In view of the decomposition of the free convolution in terms of s-free convolutions of Theorem 8.2, the result given below shows that the distributions of ω under Ψ_1 and Ψ_2 are deformations of the free convolution of semicircle laws.

Proposition 8.2. *If $(\omega_{p,q})$ is a square two-dimensional array with no zero operators, then the distributions of ω under Ψ_1 and Ψ_2 , respectively, take the form*

$$\begin{aligned}\mu_1 &= T_{1/t}(\varsigma_{\alpha,\beta} \boxplus \varsigma_{\delta,\gamma}) \uplus (\varsigma_{\delta,\gamma} \boxplus \varsigma_{\alpha,\beta}) \\ \mu_2 &= T_{1/s}(\varsigma_{\delta,\gamma} \boxplus \varsigma_{\alpha,\beta}) \uplus (\varsigma_{\alpha,\beta} \boxplus \varsigma_{\delta,\gamma})\end{aligned}$$

respectively, where $\alpha = b_{1,1}$, $\beta = b_{1,2}$, $\gamma = b_{2,1}$, $\delta = b_{2,2}$, and $t = (\beta/\alpha)^2$, $s = (\gamma/\delta)^2$.

Proof. These formulas were proven in [22, Theorem 8.1] on the basis of the continued fraction representation of the Cauchy transforms of Theorem 8.1. \blacksquare

If some of the matricially free Gaussian operators in the two-dimensional array vanish, which can be taken into account by setting the corresponding distributions to be δ_0 , their matricially free convolutions can also be retrieved from [22]. Let us analyze one special case in more detail.

Example 8.2. The limit distribution of Example 5.2, derived from Theorem 5.1 in a purely combinatorial fashion, can be obtained using convolutions. In this case, μ is the matricially free convolution of two Bernoulli distributions, $\mu_{p,q} = \kappa_{p,q}$ (concentrated at $\pm\alpha_{p,q}$) and $\mu_{q,p} = \kappa_{q,p}$ (concentrated at $\pm\alpha_{q,p}$) and thus, by [22, Theorem 7.1], it is equal to their s-free convolution

$$\boxplus_j \mu_j = \kappa_{p,q} \boxplus \kappa_{q,p}$$

which is the distribution ϑ corresponding to $a = b_{p,q}$ and $b = b_{q,p}$. Moreover, it can be seen from Theorem 6.1 that a similar result holds for the sum

$$T_{p,q}(n) = \sum_{u \in \mathcal{U}} T_{p,q}(u, n)$$

since we first compute the boolean convolution of Bernoulli laws to get

$$\kappa_{p,q} = \bigsqcup_{u \in \mathcal{U}} \kappa_{p,q}(u),$$

the Bernoulli law concentrated at $\pm\alpha_{p,q}$ of Proposition 8.1. In a similar way we obtain $\kappa_{q,p}$. Then, using matricial freeness, we obtain again the s-free convolution of two collective Bernoulli laws, $\kappa_{p,q}$ and $\kappa_{q,p}$. In other words, it suffices to replace the pair $(b_{p,q}, b_{q,p})$ by the pair $(\sum_{u \in \mathcal{U}} b_{p,q}(u), \sum_{u \in \mathcal{U}} b_{q,p}(u))$. For other matricially free convolutions, see [22, Corollary 8.2].

Finally, let us present a matricial approach to the limit distributions of Theorem 8.1. This approach leads to matricial analogs of Catalan numbers which are matrix-valued coefficients in the Laurent series for the matrix consisting of diagonal K-transforms.

Note that the Cauchy transforms of Theorem 8.1 can be expressed in terms of the K-transforms $K_{p,q}(z)$ of Corollary 8.1. Let us suppose that $b_{q,q} > 0$ for any $q \in [r]$. In that case, these transforms can be reduced to the diagonal ones. For that reason, introduce diagonal matrices

$$\begin{aligned}\mathcal{C}(z) &= \text{diag}(K_{1,1}(z), \dots, K_{r,r}(z)) \\ \mathcal{B} &= \text{diag}(b_{1,1}, \dots, b_{r,r})\end{aligned}$$

as well as the matrix

$$T = (t_{p,q}) \in M_r(\mathbb{C}), \quad \text{where } t_{p,q} = b_{p,q}/b_{p,p}$$

for any $p, q \in [r]$. Moreover, let

$$\mathcal{D} : M_r(\mathbb{C}) \rightarrow M_r(\mathbb{C})$$

be the mapping given by

$$\mathcal{D}(A) = \text{diag} \left(\sum_{i=1}^r a_{i,1}, \dots, \sum_{i=1}^r a_{i,r} \right)$$

for any $A = (a_{i,j}) \in M_r(\mathbb{C})$.

Using these notations, we can find a formula for $\mathcal{C}(z)$, from which the K-transforms of μ_q and of $\mu = \sum_q d_q \mu_q$ can be retrieved. The coefficients of the matrix-valued series obtained in this fashion play the role of matricial analogs of Catalan numbers called *Catalan matrices*.

Theorem 8.3. *If $b_{q,q} > 0$ for any $q \in [r]$, then $\mathcal{C}(z)$ assumes the form of a formal Laurent series*

$$\mathcal{C}(z) = \sum_{n=0}^{\infty} \mathcal{C}_n z^{-2n-1}$$

where (\mathcal{C}_n) is a sequence of diagonal matrices satisfying the recurrence formula

$$\mathcal{C}_n = \sum_{i+j=n-1} \mathcal{D}(\mathcal{C}_i T \mathcal{C}_j)$$

for any natural n , with $\mathcal{C}_0 = \mathcal{B}$. The series $\mathcal{C}(z)$ converges in the supremum norm for $|z| > 2\sqrt{r} \|T\| \|B\|$.

Proof. By Corollary 8.1, we have

$$K_{p,q}(z) = t_{p,q} K_{p,p}(z)$$

since $K_{p,q}(z)$ has the same denominator as $K_{p,p}(z)$ for any p, q (of course, $t_{q,q} = 1$ for any $q \in [r]$). Moreover,

$$K_{q,q}(z) = \frac{b_{q,q}}{z - \sum_{p=1}^r t_{p,q} K_{p,p}(z)}$$

for any $q \in [r]$. This leads to the equation

$$\mathcal{D}(\mathcal{C}(z) T \mathcal{C}(z)) - z \mathcal{C}(z) + B = 0$$

Since each $K_{q,q}(z)$ is the K-transform of some probability measure on the real line by Corollary 8.1, the matrix $\mathcal{C}(z)$ is of the desired form

$$\mathcal{C}(z) = \sum_{n=0}^{\infty} \mathcal{C}_n z^{-2n-1}$$

where \mathcal{C}_n is a constant diagonal matrix for each n . Substituting this series into the above equation and computing its coefficients, we obtain the desired recurrence formula. As for the convergence of the series $\mathcal{C}(z)$, note that the recurrence formula is of the same type as for Catalan numbers and thus the expression for \mathcal{C}_n has C_n terms. Using this recurrence, we can estimate the supremum norm of each \mathcal{C}_n :

$$\|\mathcal{C}_n\| \leq C_n r^n \|T\|^n \|B\|^{n+1}$$

and, denoting the right hand side by a_n , we obtain

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = \frac{1}{4r \|T\| \|B\|}$$

and thus, using D'Alembert's test for convergence, we obtain convergence for

$$|z| > 2\sqrt{r \|T\| \|B\|},$$

which completes the proof. ■

Remark 8.2. In order to see the connection between Catalan matrices and Catalan numbers in a more explicit way, let us introduce the bilinear mapping

$$\langle \cdot, \cdot \rangle : M_r(\mathbb{C}) \times M_r(\mathbb{C}) \rightarrow M_r(\mathbb{C})$$

by the formula

$$\langle A_1, A_2 \rangle = \mathcal{D}(A_1 T A_2).$$

Then, we can write the first few matrices \mathcal{C}_n in the form

$$\begin{aligned} \mathcal{C}_1 &= \langle B, B \rangle \\ \mathcal{C}_2 &= \langle B, \langle B, B \rangle \rangle + \langle \langle B, B \rangle, B \rangle \\ \mathcal{C}_3 &= \langle B, \langle B, \langle B, B \rangle \rangle \rangle + \langle B, \langle \langle B, B \rangle, B \rangle \rangle + \langle \langle B, B \rangle, \langle B, B \rangle \rangle \\ &\quad + \langle \langle B, \langle B, B \rangle \rangle, B \rangle + \langle \langle \langle B, B \rangle, B \rangle, B \rangle, \end{aligned}$$

etc. If $r = 1$ and $b_{1,1} = 1$, then \mathcal{C}_n is equal to the n th Catalan number C_n since in one of the many counting problems involving Catalan numbers, C_n is the number of different ways $n + 1$ factors can be completely parenthesized. In turn, if $b_{p,q} = d_p$ for any $p, q \in [r]$, then

$$\mathcal{C}_n = \text{diag}(d_1 C_n, \dots, d_r C_n)$$

for any natural n , which corresponds in the random matrix context to the situation in which all block variances are equal to one.

9. NON-HERMITIAN GAUSSIAN RANDOM MATRICES

We would like to generalize the results of Section 5 to the ensemble of independent non-Hermitian Gaussian random matrices. This ensemble (or, the family of its entries) is sometimes called the Ginibre Ensemble (the family of symmetric blocks of the Ginibre Ensemble can be called the Ginibre Symmetric Block Ensemble). We keep the same settings for blocks as in Section 5.

We will show that the Ginibre Symmetric Block Ensemble converges in *-moments as $n \rightarrow \infty$ to the ensemble of non-self adjoint operators

$$\{\eta_{p,q}(u) : (p, q) \in \mathcal{J}, u \in \mathcal{U}\}$$

where

$$\eta_{p,q}(u) = \widehat{\wp}_{p,q}(2u - 1) + \widehat{\wp}_{p,q}^*(2u)$$

for $(p, q) \in \mathcal{J}$, $u \in \mathcal{U} = [t]$ and $(\widehat{\wp}_{p,q}(u))$, where $u \in [2t]$, are arrays of symmetrized creation operators (the set \mathcal{J} is assumed to be symmetric). In fact, it suffices to consider the case when $\mathcal{J} = [r] \times [r]$ and restrict to a symmetric subset if needed.

In order to define t arrays $(\eta_{p,q}(u))$, where $u \in [t]$, we need $2t$ arrays $(\hat{\wp}_{p,q}(u))$, where $u \in [2t]$. Moreover, we shall assume that

$$b_{p,q}(2u-1) = b_{p,q}(2u) = d_p v_{p,q}(u)$$

for any fixed p, q, u . Our definition parallels that in the free case, where operators

$$\eta(u) = \ell(2u-1) + \ell^*(2u)$$

for $u \in [t]$, which are unitarily equivalent to circular operators, are introduced [32, Theorem 3.3]. Here, $\{\ell(1), \dots, \ell(2t)\}$ is a family of free creation operators.

We can now generalize the result of Theorem 5.1 to the Ginibre Symmetric Block Ensemble. The important difference is that we do not assume that the matrices are Hermitian and thus each matrix contains $2n^2$ independent Gaussian random variables. However, in order to reduce this model to the Hermitian case, we need to assume that the variance matrices are symmetric. The theorem given below is a block refinement of that proved by Voiculescu for the Ginibre Ensemble [32, Theorem 3.3] except that we assume the Gaussian variables to be independent block-identically distributed (i.b.d.) instead of i.i.d.

Theorem 9.1. *Let $(Y(u, n))$ be the family of complex random matrices, where $n \in \mathbb{N}$, $u \in \mathcal{U} = [t]$, and let*

$$T_{p,q}(u, n) = \sum_{(i,j) \in N_{p,q}} Y_{i,j}(u, n) \otimes e_{i,j}(n),$$

where $N_{p,q} := (N_p \times N_q) \cup (N_q \times N_p)$ for $p, q \in [r]$ and $r \in \mathbb{N}$, and where

$$\{\operatorname{Re} Y_{i,j}(u, n), \operatorname{Im} Y_{i,j}(u, n) : i, j \in [n], u \in \mathcal{U}\}$$

is a family of independent Gaussian random variables with zero mean and variances

$$\mathbb{E}((\operatorname{Re} Y_{i,j}(u, n))^2) = \mathbb{E}((\operatorname{Im} Y_{i,j}(u, n))^2) = \frac{v_{p,q}(u)}{2n}$$

for any $(i, j) \in N_{p,q}$ and $u \in \mathcal{U}$, where the matrices $(v_{p,q}(u))$ are symmetric. Then

$$\lim_{n \rightarrow \infty} \tau_q(n) (T_{p_1, q_1}^{\epsilon_1}(u_1, n) \dots T_{p_m, q_m}^{\epsilon_m}(u_m, n)) = \Psi_q(\eta_{p_1, q_1}^{\epsilon_1}(u_1) \dots \eta_{p_m, q_m}^{\epsilon_m}(u_m))$$

for any $\epsilon_1, \dots, \epsilon_m \in \{1, *\}$, $u_1, \dots, u_m \in \mathcal{U}$ and $q, p_1, q_1, \dots, p_m, q_m \in [r]$.

Proof. The proof is similar to that in [32, Theorem 3.3]. Write each matrix in the form

$$Y(u, n) = 2^{-1/2} \left(X(u, n) + i\tilde{X}(u, n) \right),$$

where the matrices

$$\begin{aligned} X(u, n) &= 2^{-1/2} (Y(u, n) + Y^*(u, n)) \\ \tilde{X}(u, n) &= i2^{-1/2} (Y^*(u, n) - Y(u, n)) \end{aligned}$$

are Hermitian for any $n \in \mathbb{N}$ and $u \in \mathcal{U}$. The symmetric blocks of these matrices, denoted $(U_{p,q}(u, n))$ and $(\tilde{U}_{p,q}(u, n))$, respectively, will give the asymptotics of symmetric blocks $(T_{p,q}(u, n))$ of $Y(u, n)$ and their adjoints since

$$T_{p,q}(u, n) = 2^{-1/2} (U_{p,q}(u, n) + i\tilde{U}_{p,q}^*(u, n))$$

for any p, q, n, u . They are built from variables $X_{i,j}(u, n)$ and $\tilde{X}_{i,j}(u, n)$, where

$$\operatorname{Re} X_{i,j}(u, n) = 2^{-1/2} (\operatorname{Re} Y_{i,j}(u, n) + \operatorname{Re} Y_{j,i}(u, n))$$

$$\begin{aligned}
\operatorname{Im} X_{i,j}(u, n) &= 2^{-1/2} (\operatorname{Im} Y_{i,j}(u, n) - \operatorname{Im} Y_{j,i}(u, n)) \\
\operatorname{Re} \tilde{X}_{i,j}(u, n) &= 2^{-1/2} (\operatorname{Im} Y_{i,j}(u, n) + \operatorname{Im} Y_{j,i}(u, n)) \\
\operatorname{Im} \tilde{X}_{i,j}(u, n) &= 2^{-1/2} (\operatorname{Re} Y_{j,i}(u, n) - \operatorname{Re} Y_{i,j}(u, n)),
\end{aligned}$$

for any $(i, j) \in N_{p,q}$ and any $u \in \mathcal{U}$. These satisfy the assumptions of Theorem 5.1. In particular, they are independent due to the fact that for fixed $i \neq j$ and u, n , the pairs $\{\operatorname{Re} Y_{i,j}(u, n), \operatorname{Re} Y_{j,i}(u, n)\}$ and $\{\operatorname{Im} Y_{i,j}(u, n), \operatorname{Im} Y_{j,i}(u, n)\}$ are identically distributed since the variance matrices $(v_{p,q}(u))$ are symmetric by assumption. Denote

$$U_{p,q}(u, n) = Z_{p,q}(2u - 1, n) \quad \text{and} \quad \tilde{U}_{p,q}(u, n) = Z_{p,q}(2u, n)$$

for any $p, q \in [r]$ and $u \in \mathcal{U}$. Using Theorem 5.1, we can express the asymptotic mixed moments in the blocks $(Z_{p,q}(u, n))$ in terms of mixed moments in symmetrized Gaussian operators $(\hat{\omega}_{p,q}(u))$, where $u \in [2t]$, namely

$$\lim_{n \rightarrow \infty} \tau_q(n)(Z_{p_1, q_1}(u_1, n) \dots Z_{p_m, q_m}(u_m, n)) = \Psi_q(\hat{\omega}_{p_1, q_1}(u_1) \dots \hat{\omega}_{p_m, q_m}(u_m))$$

for any $(p_1, q_1), \dots, (p_m, q_m) \in \mathcal{I}$, $q \in [r]$, and $u_1, \dots, u_m \in [2t]$. Each of these arrays has semicircle distributions on the diagonal and Bernoulli distributions elsewhere. These arrays are independent in the sense discussed in Section 3. Now, the linear map

$$\chi : M_r^{2t}(\mathbb{C}) \rightarrow M_r^{2t}(\mathbb{C})$$

such that

$$\begin{aligned}
\chi(e_{p,q}(2u - 1)) &= 2^{-1/2}(e_{p,q}(2u - 1) + ie_{p,q}(2u)) \\
\chi(e_{p,q}(2u)) &= 2^{-1/2}(e_{p,q}(2u - 1) - ie_{p,q}(2u))
\end{aligned}$$

for any $p, q \in [r]$ and $u \in [t]$, induces a unitary map

$$\mathcal{M}(\chi) : \mathcal{M}(\hat{\mathcal{H}}) \rightarrow \mathcal{M}(\hat{\mathcal{H}})$$

where $\mathcal{M}(\hat{\mathcal{H}})$ is the matricially free Fock space of tracial type over the array $\hat{\mathcal{H}} = (\mathcal{H}_{p,q})$ of Hilbert spaces

$$\mathcal{H}_{p,q} = \bigoplus_{u=1}^{2t} \mathbb{C}_{p,q}(u),$$

where $p, q \in [r]$, for which it holds that

$$\begin{aligned}
\mathcal{M}(\chi)\Omega_q &= \Omega_q, \\
\mathcal{M}(\chi)\wp_{p,q}(2u - 1)\mathcal{M}(\chi)^* &= 2^{-1/2}(\wp_{p,q}(2u - 1) + i\wp_{p,q}(2u)), \\
\mathcal{M}(\chi)\wp_{p,q}(2u)\mathcal{M}(\chi)^* &= 2^{-1/2}(\wp_{p,q}(2u - 1) - i\wp_{p,q}(2u))
\end{aligned}$$

for any $p, q \in [r]$ and $u \in [t]$. Consequently,

$$\mathcal{M}(\chi)\eta_{p,q}(u)\mathcal{M}(\chi)^* = 2^{-1/2}(\hat{\omega}_{p,q}(2u - 1) + i\hat{\omega}_{p,q}(2u))$$

for any $p, q \in [r]$ and $u \in [t]$. Therefore, the mixed $*$ -moments in the operators

$$\hat{\omega}_{p,q}(2u - 1) + i\hat{\omega}_{p,q}(2u)$$

can be expressed as the corresponding mixed $*$ -moments in the operators $\eta_{p,q}(u)$, where $p, q \in [r]$ and $u \in [t]$, respectively. This completes the proof. \blacksquare

Remark 9.1. Let us observe that the family of arrays

$$\{[\zeta(u)] : u \in \mathcal{U}\},$$

where each entry of $[\zeta(u)]$ is an operator of the form

$$\zeta_{p,q}(u) = \widehat{\omega}_{p,q}(2u - 1) + i\widehat{\omega}_{p,q}(2u),$$

for any $(p, q) \in \mathcal{J}$, is a matrix-valued analog of the family of Voiculescu's circular operators [32] related to the circular law. The origins of the circular law in random matrix theory date back to the work of Ginibre [14], whereas the papers of Bai [3] and Tao and Vu [31] showed its universality (see also the survey paper of Bordenave and Chafaï [9]). In particular, it is easy to see that each operator $\zeta_{p,q}(u)$ is of the form $a + ib$, where a and b are free with respect to the corresponding state Ψ_q and have semicircle distributions under Ψ_q . If $p = q$, the distributions of a and b are identical, whereas if $p \neq q$, they are identical if and only if $d_p = d_q$. The arrays $[\zeta(u)]$ play the role of *matricial circular operators*.

Theorem 9.1 can be applied to the study of asymptotic distributions of Wishart matrices and related products of complex rectangular Gaussian random matrices. For that purpose, we shall give a combinatorial formula for the mixed *-moments of the operators from the arrays $(\eta_{p,q}(u))$, where $u \in \mathcal{U}$ (in combinatorics, these operators are more convenient than those from $(\zeta_{p,q}(u))$). First, however, let us present a simple example with explicit computations of such *-moments.

Example 9.1. For simplicity, assume that $t = 1$ and denote $\eta_{p,q} = \eta_{p,q}(1)$ for any $p, q \in [r]$. Of course,

$$\eta_{p,q} = \widehat{\wp}_{p,q}(1) + \widehat{\wp}_{p,q}^*(2) \quad \text{and} \quad \eta_{p,q}^* = \widehat{\wp}_{p,q}^*(1) + \widehat{\wp}_{p,q}(2)$$

For any $q \in [r]$, we have

$$\begin{aligned} \Psi_q(\eta_{p,q}\eta_{p,q}^*) &= \Psi_q(\wp_{p,q}^*(2)\wp_{p,q}(2)) = b_{p,q}(2) \\ \Psi_q((\eta_{p,q}\eta_{p,q}^*)^2) &= \Psi_q(\wp_{p,q}^*(2)\wp_{p,q}(2)\wp_{p,q}^*(2)\wp_{p,q}(2)) \\ &\quad + \Psi_q(\wp_{p,q}^*(2)\wp_{q,p}^*(1)\wp_{q,p}(1)\wp_{p,q}(2)) \\ &= b_{p,q}^2(2) + b_{p,q}(2)b_{q,p}(1) \\ \Psi_q((\eta_{p,q}\eta_{p,q}^*)^3) &= \Psi_q(\wp_{p,q}^*(2)\wp_{p,q}(2)\wp_{p,q}^*(2)\wp_{p,q}(2)\wp_{p,q}^*(2)\wp_{p,q}(2)) \\ &\quad + \Psi_q(\wp_{p,q}^*(2)\wp_{p,q}(2)\wp_{p,q}^*(2)\wp_{q,p}^*(1)\wp_{q,p}(1)\wp_{p,q}(2)) \\ &\quad + \Psi_q(\wp_{p,q}^*(2)\wp_{q,p}^*(1)\wp_{q,p}(1)\wp_{p,q}(2)\wp_{p,q}^*(2)\wp_{p,q}(2)) \\ &\quad + \Psi_q(\wp_{p,q}^*(2)\wp_{q,p}^*(1)\wp_{q,p}(1)\wp_{q,p}^*(1)\wp_{q,p}(1)\wp_{p,q}(2)) \\ &\quad + \Psi_q(\wp_{p,q}^*(2)\wp_{q,p}^*(1)\wp_{p,q}^*(2)\wp_{p,q}(2)\wp_{q,p}(1)\wp_{p,q}(2)) \\ &= b_{p,q}^3(2) + 3b_{p,q}^2(2)b_{q,p}(1) + b_{p,q}(2)b_{q,p}^2(1) \end{aligned}$$

and thus the summands correspond to non-crossing colored partitions. For instance, the summands in the last moment correspond to suitably colored partitions shown in Example 5.4. In the random matrix setting of Theorem 9.1, if all block variances are set to one, the expressions on the right-hand sides reduce to polynomials in d_p, d_q of the form

$$d_p, \quad d_p^2 + d_p d_q, \quad d_p^3 + 3d_p^2 d_q + d_p d_q^2$$

respectively, since $b_{p,q}(k) = d_p$ and $b_{q,p}(k) = d_q$ for $k \in \{1, 2\}$ (cf., for instance, [16]).

The above example shows the connection between mixed $*$ -moments of the considered operators and non-crossing colored partitions. The main feature of the combinatorics for the non-Hermitian case is that in order to get a pairing between two operators, one of them has to be starred and one unstarred. This leads to the following definition and, consequently, to an analog of Proposition 4.3.

Definition 9.1. We say that $\pi \in \mathcal{NC}_m^2$ is *adapted* to $((w_1, u_1, \epsilon_1), \dots, (w_m, u_m, \epsilon_m))$, where $w_k = \{p_k, q_k\}$ and $(p_k, q_k, u_k) \in [r] \times [r] \times [t]$ and $\epsilon_k \in \{1, *\}$ for any k , if there exists a tuple $((v_1, u_1), \dots, (v_m, u_m))$, where $v_k \in \{(p_k, q_k), (q_k, p_k)\}$ for any k , to which π is adapted and $\epsilon_j \neq \epsilon_k$ whenever $\{j, k\}$ is a block. The set of such partitions will be denoted by $\mathcal{NC}_m^2((w_1, u_1, \epsilon_1), \dots, (w_m, u_m, \epsilon_m))$. Its subset consisting of those partitions which are in $\mathcal{NC}_{m,q}^2((w_1, u_1), \dots, (w_m, u_m))$ will be denoted $\mathcal{NC}_{m,q}^2((w_1, u_1, \epsilon_1), \dots, (w_m, u_m, \epsilon_m))$.

Proposition 9.1. *For any tuple $((w_1, u_1, \epsilon_1), \dots, (w_m, u_m, \epsilon_m))$ and $q \in [r]$, $m \in \mathbb{N}$, where $w_k = \{p_k, q_k\}$ and $p_k, q_k \in [r]$, $u_k \in [t]$ and $\epsilon_k \in \{1, *\}$ for each k , it holds that*

$$\Psi_q(\eta_{p_1, q_1}^{\epsilon_1}(u_1) \dots \eta_{p_m, q_m}^{\epsilon_m}(u_m)) = \sum_{\pi \in \mathcal{NC}_{m,q}^2((w_1, u_1, \epsilon_1), \dots, (w_m, u_m, \epsilon_m))} b_q(\pi, f)$$

where f is the coloring of π defined by $((w_1, u_1), \dots, (w_m, u_m))$ and by the number q .

Proof. The proof is similar to that of Proposition 4.3. As in the case of $\hat{\omega}_{p,q}(u)$, each operator $\eta_{p,q}(u)$ and $\eta_{p,q}^*(u)$ is a sum of a symmetrized creation operator and a symmetrized annihilation operator. However, in the non-self-adjoint case, these summands are labelled by two different numbers, $2u-1$ and $2u$, respectively. Therefore, the non-crossing pairings can be of two types: $(\eta_{p,q}(u), \eta_{p,q}^*(u))$ or $(\eta_{p,q}^*(u), \eta_{p,q}(u))$, which means that the associated non-crossing partitions must satisfy the additional condition $\epsilon_j \neq \epsilon_k$ whenever $\{j, k\}$ is a block. This completes the proof. \blacksquare

10. WISHART MATRICES AND RELATED PRODUCTS

The main point of this section is to show that the framework of matricial freeness is general enough to include sums and products of independent Gaussian random matrices with block-identical variances in one unified scheme, although the formalism is much more general and should give other interesting examples.

In particular, we can reproduce certain results concerning products $B(n)$ of rectangular Gaussian random matrices and the asymptotic distributions of $B(n)B^*(n)$ under the trace composed with classical expectation as $n \rightarrow \infty$. In particular, if $B(n)$ is just one Gaussian random matrix, the matrix

$$W(n) = B(n)B^*(n)$$

is the *complex Wishart matrix* [37]. The limit distribution of a sequence of such matrices is the Marchenko-Pastur distribution (which also plays the role of the free Poisson distribution) with moments given by Catalan numbers.

The original result on the asymptotic distributions of the Wishart matrices is due to Marchenko and Pastur [25], but many different proofs have been given (see, for instance [15, 27, 35]). In the case when $B(n)$ is a power of a square random matrix, it has recently

been shown by Alexeev *et al* [1] that the limit moments are given by Fuss-Catalan numbers (products of independent random matrices have also been studied recently [2,10]). The distributions defined by the Fuss-Catalan numbers were explicitly determined by Penson and Życzkowski [29]. A random matrix model based on a family of distributions called free Bessel laws constructed from the Marchenko-Pastur distribution by means of free convolutions was given by Banica *et al* [4]. In turn, asymptotic freeness of independent Wishart matrices was proved by Voiculescu [32] (see also the stronger versions of Capitaine and Casalis [11] and Hiai and Petz [18]). Let us also mention that explicit formulas for moments of Wishart matrices were given, for instance, by Hanlon *et al* [17] and by Haagerup and Thorbjørnsen [16].

In our framework, it suffices to use one symmetric off-diagonal (rectangular or square, balanced or unbalanced) block to study the asymptotic distribution of the Wishart matrix. More generally, using the same operator algebras as in Section 5, we can also study the asymptotic distributions of the matrices $W(n)$ in the case when each $B(n)$ is a sum of independent rectangular Gaussian random matrices as in [6], namely

$$B(n) = Y(u_1, n) + Y(u_2, n) + \dots + Y(u_m, n),$$

assuming they have the same dimensions, as well as in the case when $B(n)$ is a product of independent rectangular Gaussian random matrices

$$B(n) = Y(u_1, n)Y(u_2, n) \dots Y(u_k, n),$$

assuming their dimensions are such that the products are well-defined. Using Theorems 5.1 and 9.1, one can study non-Hermitian Wishart matrices as well. In both cases, it suffices to take a suitable sequence of the off-diagonal symmetric blocks of a family of independent Gaussian random matrices (the first case) or even one Gaussian random matrix matrix (the second case). Since we know how to compute the asymptotic joint distributions of symmetric blocks, we can immediately obtain results concerning asymptotic distributions of such products.

In our treatment of products, it will be convenient to use sets $\mathcal{NC}_m^2(W_{k,p})$ of non-crossing $W_{k,p}$ -pairings, where $W_{k,p}$ is a word of the form

$$W_{k,p} = (12 \dots pp^* \dots 2^*1^*)^k$$

where $m = 2kp$, by which we understand the set of non-crossing pair partitions of the set $[m]$ in which all blocks are associated with pairs of letters of the form $\{j, j^*\}$. If $\pi \in \mathcal{NC}_m^2(W_{k,p})$, we denote by $\mathcal{R}_j(\pi)$ and $\mathcal{R}_j^*(\pi)$ the sets of the right legs of π which are associated with j and j^* , respectively.

Definition 10.1. Define homogenous polynomials in variables d_1, d_2, \dots, d_{p+1} of the form

$$P_k(d_1, d_2, \dots, d_{p+1}) = \sum_{\pi \in \mathcal{NC}_{2kp}^2(W_{k,p})} d_1^{r_1(\pi)} d_2^{r_2(\pi)} \dots d_{p+1}^{r_{p+1}(\pi)}$$

for any $k, p \in \mathbb{N}$, where

$$r_j(\pi) = |\mathcal{R}_j(\pi)| + |\mathcal{R}_{j-1}^*(\pi)|$$

for any $\pi \in \mathcal{NC}_{2kp}^2(W)$ and $1 \leq j \leq p+1$, where we set $\mathcal{R}_0^*(\pi) = \emptyset$ and $\mathcal{R}_{p+1}(\pi) = \emptyset$.

Our random matrix model for products of independent Gaussian random matrices will be based on products of a chain of symmetric random blocks

$$T_{j,j+1}(n) = T_{j,j+1}(u, n), \text{ where } j \in [p] \text{ and } n \in \mathbb{N},$$

with fixed $u \in \mathcal{U}$ and thus omitted in our notations (taking different u 's does not change the computations since these blocks are independent anyway). These blocks are embedded in a Gaussian random matrix $Y(n)$ with asymptotic dimensions d_1, d_2, \dots, d_{p+1} which will be the variables of our polynomials, namely

$$Y(n) = \sum_{j=1}^p T_{j,j+1}(n)$$

where each $T_{j,j+1}(n)$ consists of two blocks, $S_{j,j+1}(n)$ and $S_{j+1,j}(n)$, placed symmetrically with respect to the main diagonal. The blocks $T_{j,j+1}(n)$ are not assumed to be Hermitian, thus we shall use Theorem 9.1 when discussing the limit distribution of $W(n)$. Using products of Hermitian random blocks is also possible, especially in the simplest case of Wishart matrices. However, it is more convenient to use products of non-Hermitian symmetric random blocks to give a combinatorial description of the limit moments of $W(n)$ in the case when $B(n)$ is a product of order higher than two.

In the proof given below, we use ideas on the enumeration of non-crossing pairings given by Kemp and Speicher [20] (related to the results in the book of Nica and Speicher [26]).

Theorem 10.1. *Under the assumptions of Theorem 9.1, suppose that \mathcal{U} consists of one element and $v_{j,j+1} = 1$ for $j \in [p]$, where p is a fixed natural number, and let*

$$B(n) = T_{1,2}(n)T_{2,3}(n) \dots T_{p,p+1}(n)$$

for any $n \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \tau_1(n) \left((B(n)B^*(n))^k \right) = P_k(d_1, d_2, \dots, d_{p+1})$$

where d_1, d_2, \dots, d_{p+1} are asymptotic dimensions.

Proof. By Theorem 9.1, we obtain

$$\lim_{n \rightarrow \infty} \tau_1(n) ((B(n)B^*(n))^k) = \Psi_1((\eta\eta^*)^k)$$

for any natural k , where

$$\eta = \eta_{1,2}\eta_{2,3} \dots \eta_{p,p+1}$$

for any given p . In view of Proposition 9.1, we can express the right-hand side in terms of non-crossing partitions

$$\Psi_1((\eta\eta^*)^k) = \sum_{\pi \in \mathcal{NC}_{m,1}^2((w_1, \epsilon_1), \dots, (w_m, \epsilon_m))} b_1(\pi, f)$$

where $m = 2kp$ and

$$\begin{aligned} (w_{1+2jp}, \dots, w_{p+2jp}) &= (\{1, 2\}, \dots, \{p, p+1\}) \\ (w_{p+1+2jp}, \dots, w_{2p+2jp}) &= (\{p, p+1\}, \dots, \{1, 2\}) \end{aligned}$$

for any $0 \leq j \leq k-1$. In the notation for non-crossing pair partitions adapted to $((w_1, \epsilon_1), \dots, (w_m, \epsilon_m))$ we omit u_1, \dots, u_m since in the considered case we have one array $(\eta_{i,j})$. Each $\eta_{j,j+1}$ is a sum of the form

$$\eta_{j,j+1} = \hat{\phi}_{j,j+1}(1) + \hat{\phi}_{j,j+1}^*(2),$$

but in the computation of moments, the information about which creation-annihilation pairs (labelled by 1 or 2) contribute to the moment $\Psi_1((\eta\eta^*)^k)$ is encoded in $\epsilon =$

$(\epsilon_1, \dots, \epsilon_m)$. Namely, the pairs $(\hat{\wp}_{j,j+1}^*(1), \hat{\wp}_{j,j+1}(1))$ or $(\hat{\wp}_{j,j+1}^*(2), \hat{\wp}_{j,j+1}(2))$ contribute a number from the set $\{b_{j,j+1}, b_{j+1,j}\} = \{d_j, d_{j+1}\}$ depending on whether $(\eta_{j,j+1}^*, \eta_{j,j+1})$ or $(\eta_{j,j+1}, \eta_{j,j+1}^*)$ is associated with the given block of π . Moreover, only operators with the same matricial indices can form a pairing and thus $\mathcal{NC}_{m,1}^2((w_1, \epsilon_1), \dots, (w_m, \epsilon_m))$ can be put in one-to-one correspondence with the set $\mathcal{NC}_{2kp}^2(W_{k,p})$. If now $\pi \in \mathcal{NC}_{2kp}^2(W_{k,p})$, then we will identify $b_1(\pi, f)$ with the corresponding expression of Proposition 9.1. Recall that

$$b_1(\pi, f) = \prod_{j=1}^{kp} b_1(\pi_j, f)$$

where $\{\pi_1, \dots, \pi_{kp}\}$ are blocks of π and where the coloring f is uniquely determined by π and by the coloring of the imaginary block (in this case, the latter is equal to one by our choice of $\tau_1(n)$). Since the block variances are set to one, we can express each $b_1(\pi, f)$ as a monomial of order $m = kp$ in asymptotic dimensions of the form

$$b_1(\pi, f) = d_1^{r_1(\pi)} d_2^{r_2(\pi)} \dots d_{p+1}^{r_{p+1}(\pi)}$$

where the exponents are natural numbers which depend on π and for which

$$r_1(\pi) + r_2(\pi) + \dots + r_{p+1}(\pi) = kp.$$

Let us find an explicit combinatorial formula for these numbers. When computing $b_1(\pi_k, f)$ for various blocks of π , we can see that asymptotic dimensions are assigned to the right legs associated with letters $\{j, j^* : j \in [p]\}$ according to the rules

$$j \rightarrow d_j \text{ and } j^* \rightarrow d_{j+1}$$

for any $j \in [p]$. This leads to the formula

$$r_j(\pi) = |\mathcal{R}_j(\pi)| + |\mathcal{R}_{j-1}^*(\pi)| \text{ for } j \in [p+1],$$

where $\pi \in \mathcal{NC}_m^2(W)$, and thus

$$\Psi_1((\eta\eta^*)^k) = \sum_{\pi \in \mathcal{NC}_{2kp}^2(W_{k,p})} d_1^{r_1(\pi)} d_2^{r_2(\pi)} \dots d_{p+1}^{r_{p+1}(\pi)},$$

which completes the proof. ■

The special case of $p = 1$ corresponds to Wishart matrices and the Marchenko-Pastur distribution [25] with shape parameter $t > 0$, namely

$$\varrho_t = \max\{1-t, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \mathbb{1}_{[a,b]}(x)dx$$

where $a = (1 - \sqrt{t})^2$ and $b = (1 + \sqrt{t})^2$ (see Corollary 10.1). At the same time, ϱ_t is the free Poisson law in free probability. It has been shown by Oravecz and Petz [27] that its moments are the *Narayana polynomials*

$$N_k(t) = \sum_{j=1}^k N(k, j)t^j$$

for any $k \in \mathbb{N}$, with coefficients

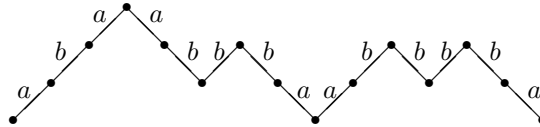
$$N(k, j) = \frac{1}{j} \binom{k-1}{j-1} \binom{k}{j-1}$$

called *Narayana numbers*. These numbers are obtained in several different enumerations. For instance, $N(k, j)$ is equal to the number of Catalan paths of length $2k$ with j peaks. However, we will use another enumeration related to right legs of blocks of the associated non-crossing partitions.

Proposition 10.1. *The Narayana number $N(k, j)$ is equal to the number of those non-crossing pair partitions $\pi \in \mathcal{NC}_{2k}^2$ which have j even numbers in the set $\mathcal{R}(\pi)$.*

Proof. Our proof is based on the enumeration derived by Osborn [28], in which $N(k, j)$ is the number of Catalan paths of length $2k$ with $2j$ edges lying in odd bands, where an odd band is a set of the form $\mathbb{R} \times [i, i + 1]$ in the plane \mathbb{R}^2 , where i is even, in the standard setting, in which a Catalan path begins at $(0, 0)$ and ends at $(2k, 0)$. Now, let us associate with each Catalan path a non-crossing pair partition $\pi \in \mathcal{NC}_{2k}^2$ in the canonical way. Then, if $\{l, r\}$ is a block of π with $l < r$, where r is even, then the edges corresponding to l and r lie in the odd band. Similarly, if r is odd, then these edges lie in the even band. Consequently, $N(k, j)$ is equal to the number of non-crossing pair partitions of $[2k]$ having j right legs which are even and $k - j$ right legs which are odd. This completes the proof. \blacksquare

Example 10.1. In the path shown below all edges labelled by a lie in the odd bands and all edges labelled by b lie in the even bands. Therefore, there are 6 edges lying in the odd bands and 8 edges lying in the even bands, thus $j = 3$.



Corollary 10.1. *If $p = 1$ and $d_1 > 0$, then under the assumptions of Theorem 10.1 we obtain*

$$\lim_{n \rightarrow \infty} \tau_1(n) \left((B(n)B^*(n))^k \right) = d_1^k N_k(d_2/d_1),$$

and thus the limit distribution is the d_1 -dilation of the Marchenko-Pastur distribution ϱ_t with the shape parameter $t = d_2/d_1$, denoted ϱ_{d_2, d_1} .

Proof. The letters 1 and 1^* correspond to odd and even numbers, respectively, if we use the notation as in the proof of Theorem 10.1. In turn, the number of non-crossing pair partitions which have j right legs labelled by 1^* is given by $N(k, j)$ by Proposition 9.1 and thus

$$P_k(d_1, d_2) = \sum_{j=1}^k N(k, j) d_2^j d_1^{k-j} = d_1^k N_k(d_2/d_1)$$

which proves the first assertion. The second one is clear since the transformation on moments $m_k \rightarrow \lambda^k m_k$ leads to the dilation of the corresponding measures. \blacksquare

Remark 10.1. The usual settings for Wishart matrices are often slightly different since different normalizations are used [10,19]. For instance, if $B(N) \in GRM(m(N), N, 1/N)$

is a Gaussian random matrix of dimension $m(N) \times N$ and $1/N$ is the variance of each entry, where $N \in \mathbb{N}$, it is assumed that

$$t = \lim_{N \rightarrow \infty} \frac{m(N)}{N}$$

and then one computes the limit distribution of $B^*(N)B(N)$ under normalized trace tr_N composed with classical expectation. One obtains Narayana polynomials in t as the limit moments, with the corresponding Marchenko-Pastur distribution with shape parameter t . In our model, we embed $B(n)$ and $B^*(n)$ in a larger square matrix of dimension n , in which $B(n) = S_{1,2}(n)$ and $B^*(n) = S_{1,2}^*(n) = S_{2,1}(n)$ are off-diagonal blocks and we compute the limit moments of $B(n)B^*(n)$ under $\tau_1(n)$. In order to directly compare these two approaches, we set $N = n_1(n)$ and $m(N) = n_2(n)$ to get $t = d_2/d_1$ and, since the variance in our approach is $1/n$ and in Marchenko-Pastur's theorem the variance is $1/N$, the k th moment in the limit moment obtained in Marchenko-Pastur's theorem must be multiplied by d_1^k to give our asymptotic product polynomial.

Example 10.2. If $p = 1$, the lowest order polynomials of Definition 10.1 are

$$\begin{aligned} P_1(d_1, d_2) &= d_2 \\ P_2(d_1, d_2) &= d_2 d_1 + d_2^2 \\ P_3(d_1, d_2) &= d_2 d_1^2 + 3d_2^2 d_1 + d_2^3 \\ P_4(d_1, d_2) &= d_2 d_1^3 + 6d_2^2 d_1^2 + 6d_2^3 d_1 + d_2^4 \end{aligned}$$

and can be obtained directly from Corollary 10.1 since the corresponding Narayana polynomials are $N_1(t) = t$, $N_2(t) = t + t^2$, $N_3(t) = t + 3t^2 + t^3$, $N_4(t) = t + 6t^2 + 6t^3 + t^4$, respectively.

Corollary 10.2. If $d_1 = d_2 = \dots = d_{p+1} = d$, then

$$P_k(d_1, d_2, \dots, d_{p+1}) = d^{kp} F(p, k)$$

where $F(p, k) = \frac{1}{pk+1} \binom{pk+k}{k}$ are Fuss-Catalan numbers.

Proof. If $d_1 = \dots = d_{p+1} = d$, then $P_k(d_1, d_2, \dots, d_{p+1})$ is equal to d^{kp} multiplied by the number of non-crossing pair partitions adapted to the word $W_{k,p}$ by Theorem 10.1. It is well-known [20] that the latter is equal to the Fuss-Catalan number $F(p, k)$. ■

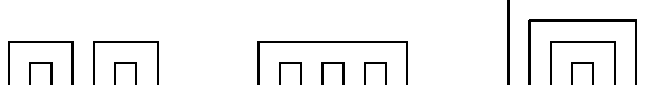
Example 10.3. If $p = 2$, we obtain by Theorem 10.1 the limit moments

$$\Psi_1((\eta\eta^*)^k) = P_k(d_1, d_2, d_3), \quad \text{where } \eta = \eta_{1,2}\eta_{2,3}.$$

For instance, contributions to $P_2(d_1, d_2, d_3)$ are

$$\begin{aligned} \Psi_1(\wp_{2,1}^* \wp_{3,2}^* \wp_{3,2} \wp_{2,1} \wp_{2,1}^* \wp_{3,2}^* \wp_{3,2} \wp_{2,1}) &= d_2^2 d_3^2 \\ \Psi_1(\wp_{2,1}^* \wp_{3,2}^* \wp_{3,2} \wp_{1,2}^* \wp_{1,2} \wp_{3,2}^* \wp_{3,2} \wp_{2,1}) &= d_1 d_2 d_3^2 \\ \Psi_1(\wp_{2,1}^* \wp_{3,2}^* \wp_{2,3}^* \wp_{1,2}^* \wp_{1,2} \wp_{2,3} \wp_{3,2} \wp_{2,1}) &= d_1 d_2^2 d_3 \end{aligned}$$

with the corresponding partitions of the associated word $W_{2,2} = 122^*1^*122^*1^*$



respectively. Thus,

$$P_2(d_1, d_2, d_3) = d_2^2 d_3^2 + d_1 d_2 d_3^2 + d_1 d_2^2 d_3.$$

If we define multivariate generalizations of Narayana polynomials $N_k(t_1, t_2, \dots, t_p)$ by the formula

$$P_k(d_1, d_2, \dots, d_{p+1}) = d_1^{kp} N_k(d_2/d_1, d_3/d_1, \dots, d_{p+1}/d_1),$$

we obtain

$$N_2(t_1, t_2) = t_1^2 t_2^2 + t_1 t_2^2 + t_1^2 t_2.$$

These polynomials and their coefficients are of some interest from the combinatorial point of view and will be studied elsewhere in more detail.

The asymptotics of the Wishart matrix can be generalized in another direction, namely we can study the asymptotic distribution of $W(n) = B(n)B^*(n)$, where $B(n)$ is now a sum of independent rectangular Gaussian random matrices. In our theory, we can compute the corresponding limit moments under the trace by choosing each of these matrices to be the rectangular off-diagonal blocks $S_{1,2}(u, n)$ and using either Hermitian or non-Hermitian symmetric blocks containing them. In the proof of the theorem given below, we use Hermitian symmetric blocks.

Theorem 10.2. *If $\{S(u, n) : u \in \mathcal{U}\}$ is a family of rectangular independent complex Gaussian random matrices with identically distributed entries for any $n \in \mathbb{N}$ and*

$$B(n) = \sum_{u \in \mathcal{U}} S(u, n),$$

then the limit distribution of $W(n) = B(n)B^(n)$ under the partial trace $\tau_1(n)$ is ϱ_{d_2, d_1} .*

Proof. Let us treat each matrix $S(u, n)$ as the block $S_{1,2}(u, n)$ in a Hermitian random matrix $Y(u, n)$. First, let us show that

$$\lim_{n \rightarrow \infty} \tau_1(n)((B(n)B^*(n))^k) = \Psi_1((\hat{\omega}_{1,2})^{2k})$$

for any $k \in \mathbb{N}$ for some $\hat{\omega}_{1,2} = \omega_{1,2} + \omega_{2,1}$. If we embed the blocks $S_{1,2}(u, n)$ in the Hermitian blocks $T_{1,2}(u, n)$, then

$$\begin{aligned} & \tau_1(n)(S_{1,2}(u_1, n)S_{1,2}^*(u_2, n) \dots S_{1,2}(u_{2k-1}, n)S_{1,2}^*(u_{2k}, n)) \\ &= \tau_1(n)(T_{1,2}(u_1, n)T_{1,2}(u_2, n) \dots T_{1,2}(u_{2k-1}, n)T_{1,2}(u_{2k}, n)) \end{aligned}$$

for any $u_1, \dots, u_{2k} \in \mathcal{U}$ and $n \in \mathbb{N}$ since each Hermitian block $T_{1,2}(u_j, n)$ consists of two blocks, $S_{1,2}(u_j, n)$ and its adjoint $S_{1,2}^*(u_j, n)$, and the partial trace $\tau_1(n)$ selects only vectors from the set $\{e_i : i \in N_1\}$ onto which the product acts. Therefore, by Theorem 5.1, we obtain

$$\lim_{n \rightarrow \infty} \tau_1(n)((B(n)B^*(n))^k) = \Psi_1((\hat{\omega}_{1,2})^{2k})$$

where

$$\hat{\omega}_{1,2} = \sum_{u \in \mathcal{U}} \hat{\omega}_{1,2}(u)$$

and $\widehat{\omega}_{1,2} = \omega_{1,2} + \omega_{2,1}$, with $\omega_{1,2}$ and $\omega_{2,1}$ associated with numbers

$$b_{1,2} = \sum_{u \in \mathcal{U}} b_{1,2}(u) \quad \text{and} \quad b_{2,1} = \sum_{u \in \mathcal{U}} b_{2,1}(u).$$

In the computation of moments of $\widehat{\omega}_{1,2}$ under Ψ_1 , as in Example 5.2, these numbers are assigned to blocks of even and odd depths, respectively. Similar arguments to those in the proofs of Proposition 10.1 and Corollary 10.1 allow us to conclude that the moments of $\widehat{\omega}_{1,2}^2$ under Ψ_1 agree with the moments of the d_1 -dilation of the Marchenko-Pastur distribution with the shape parameter $t = b_{2,1}/b_{1,2} = d_2/d_1$ (by the definition of Hermitian blocks, we must have $v_{1,2}(u) = v_{2,1}(u)$ for any $u \in \mathcal{U}$). \blacksquare

Example 10.4. The above limit distributions are closely related to those given by the Double Convolution Formula. If $\mathcal{J} = \{(1, 2), (2, 1)\}$, then the symmetrization of the limit distribution of $W(n)$ under $\tau_1(n)$ is the distribution of the form

$$\mu = \boxtimes_{j \in \mathcal{J}} \mu_j = \kappa_{2,1} \boxplus \kappa_{1,2}$$

corresponding to Ψ_1 , where μ_j is the Bernoulli law concentrated at $\pm\sqrt{b_j}$ for $j \in \mathcal{J}$. By the symmetrization of a probability measure ν on $[0, \infty)$ with moments $\nu(n)$ we understand the symmetric probability measure ν_{sym} on \mathbb{R} with even moments $\nu_{\text{sym}}(2n) = \nu(n)$ and odd moments equal to zero. An analogous result is obtained for the limit distribution under $\tau_2(n)$ corresponding to Ψ_2 .

Remark 10.2. Alternatively, we can treat each matrix $S(u, n)$ as the block $S_{1,2}(u, n)$ in a non-Hermitian random matrix $Y(u, n)$. Then again we can take

$$B(n) = \sum_{u \in \mathcal{U}} T_{1,2}(u, n),$$

except that symmetric blocks $T_{1,2}(u, n)$ are not Hermitian. Using Theorem 9.1, we obtain

$$\lim_{n \rightarrow \infty} \tau_1(n)((B(n)B^*(n))^k) = \Psi_1((\eta_{1,2}\eta_{1,2}^*)^k),$$

for any $k \in \mathbb{N} \cup \{0\}$, where

$$\eta_{1,2} = \sum_{u \in \mathcal{U}} \eta_{1,2}(u)$$

which gives the same limit distribution as in Theorem 10.2.

Theorem 10.3. *Under the assumptions of Theorem 10.1, the family of Wishart matrices*

$$\mathcal{W}(n) = \{S(u, n)S^*(u, n) : u \in \mathcal{U}\}$$

is asymptotically free under $\tau_1(n)$.

Proof. As in the proof of Theorem 10.2, the Wishart matrices will be built from blocks $S_{1,2}(u)$ and their adjoints,

$$W(u, n) = S_{1,2}(u, n)S_{1,2}^*(u, n)$$

for any $u \in \mathcal{U}$ and $n \in \mathbb{N}$. Using Theorem 5.1 and arguments similar to those in the proof of Theorem 10.2, we can replace $S_{1,2}(u, n)$ by Hermitian $T_{1,2}(u, n)$, which gives

$$\lim_{n \rightarrow \infty} \tau_1(n)(W(u_1, n) \dots W(u_k, n)) = \Psi_1(\widehat{\omega}_{1,2}^2(u_1) \dots \widehat{\omega}_{1,2}^2(u_k))$$

for any $u_1, \dots, u_k \in \mathcal{U}$ and $n \in \mathbb{N}$. It suffices to prove that the family $\{\hat{\omega}_{1,2}^2(u) : u \in \mathcal{U}\}$ is free with respect to Ψ_1 . More generally, we will show that the family of algebras $\{\mathcal{A}(u) : u \in \mathcal{U}\}$, where

$$\mathcal{A}(u) = \mathbb{C}\langle \hat{\omega}_{1,2}^2(u), r_{1,2}, r_{2,1} \rangle$$

and $r_{1,2}, r_{2,1}$ are given by Definition 2.5 (we have $\hat{1}_{1,2} = r_{1,2} + r_{2,1}$), is free with respect to Ψ_1 . Any polynomial from $\mathcal{A}(u)$ is a linear combination of $r_{1,2}, r_{2,1}$ and even powers of $\hat{\omega}_{1,2}(u)$ interchanged with $r_{1,2}$ and $r_{2,1}$. However, using the relations

$$\wp_{1,2}^*(u)\wp_{1,2}(u) = b_{1,2}(u)r_{1,2}, \quad \wp_{2,1}^*(u)\wp_{2,1}(u) = b_{2,1}(u)r_{2,1},$$

and

$$r_{1,2}\wp_{1,2}(u) = 0, \quad r_{2,1}\wp_{1,2}(u) = \wp_{1,2}(u), \quad r_{2,1}\wp_{2,1}(u) = 0, \quad r_{1,2}\wp_{2,1}(u) = \wp_{2,1}(u),$$

we can observe that any polynomial $a_m \in \mathcal{A}(u_m) \cap \text{Ker}\Psi_1$, when acting onto Ω_1 , reduces to a linear combination of products of the form

$$(\wp_{1,2}(u_m)\wp_{2,1}(u_m))^n$$

where $n > 0$, since the remaining terms give zero (in particular, $r_{1,2}\Omega_1 = 0$). The same is true when we then act succesively with polynomials a_{m-1}, \dots, a_1 , where $a_i \in \mathcal{A}(u_i) \cap \text{Ker}\Psi_1$ and $u_1 \neq \dots \neq u_m$. We obtain a linear combination of vectors of the form

$$(e_{1,2}(u_1) \otimes e_{2,1}(u_1))^{n_1} \otimes \dots \otimes (e_{1,2}(u_m) \otimes e_{2,1}(u_m))^{n_m}$$

where $n_1 + \dots + n_m > 0$, and thus it is orthogonal to Ω_1 , which gives the freeness condition

$$\Psi_1(a_1 a_2 \dots a_m) = 0.$$

It remains to observe that $\hat{1}_{1,2}$ is the projection onto the Hilbert space

$$\mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2 \oplus \bigoplus_k (\mathcal{N}_{1,k} \oplus \mathcal{N}_{2,k}),$$

which is left invariant by all elements of each $\mathcal{A}(u)$, which completes the proof. \blacksquare

Let us finally establish a relation between the limit distributions of Theorem 10.1 and free Bessel laws $\pi_{p,t}$ of Banica *et al* [4] expressed in terms of the free multiplicative convolution

$$\pi_{p,t} = \pi^{\boxtimes(p-1)} \boxtimes \pi^{\boxplus t},$$

where $\pi = \varrho_1$ is the standard Marchenko-Pastur distribution with the shape parameter equal to one. Random matrix models for these laws given in [4] were based on the multiplication of independent Gaussian random matrices.

Let $\mu_s = U_s \mu$ be the probability measure on the real line defined by

$$G_{\mu_s}(z) = sG_\mu(z) + \frac{1-s}{z}$$

for any $s > 0$, where G_{μ_s} and G_μ are Cauchy transforms of μ_s and μ , respectively.

Theorem 10.4. *If $d_1, d_2, \dots, d_{p+1} > 0$ and $p > 1$, then the limit distribution of Theorem 10.1 takes the form*

$$\mu_1 = U_s(\varrho_{d_1,d_2} \boxtimes \varrho_{d_2,d_3} \boxtimes \dots \boxtimes \varrho_{d_{p+1},d_p})$$

where $s = d_p/d_1$.

Proof. We have

$$\begin{aligned}
& \tau_1(n) \left((S_{1,2}(n) \dots S_{p,p+1}(n) S_{p,p+1}^*(n) \dots S_{1,2}^*(n))^k \right) \\
&= n_2/n_1 \cdot \tau_2(n) \left((S_{2,3}(n) \dots S_{p,p+1}(n) S_{p,p+1}^*(n) \dots S_{2,3}^*(n) S_{1,2}^*(n) S_{1,2}(n))^k \right) \\
&= n_2/n_1 \cdot \tau_2(n) \left((T_{2,3}(n) \dots T_{p,p+1}(n) T_{p,p+1}(n) \dots T_{2,3}(n) T_{1,2}(n) T_{1,2}(n))^k \right)
\end{aligned}$$

which tends to

$$d_2/d_1 \cdot \Psi_2((\tilde{\omega}^2 \hat{\omega}_{1,2}^2)^k),$$

as $n \rightarrow \infty$, where $\tilde{\omega} = \hat{\omega}_{2,3} \dots \hat{\omega}_{p,p+1}$. Now, the pair $\{\tilde{\omega}^2, \hat{\omega}_{1,2}^2\}$ is free with respect to Ψ_2 . The proof of this fact is similar to the proof of the asymptotic freeness of independent Wishart matrices given in Theorem 10.3 and is omitted. Now, the Ψ_2 -distribution of $\hat{\omega}_{1,2}^2$ is ϱ_{d_1, d_2} by Corollary 10.1. Therefore,

$$\mu_1 = U_{d_2/d_1}(\varrho_{d_1, d_2} \boxtimes \mu_2)$$

where μ_2 is the Ψ_2 -distribution of $\tilde{\omega}^2$. If $\tilde{\omega}$ is a product of at least two operators, the same procedure is applied to μ_2 , which gives

$$\mu_2 = U_{d_3/d_2}(\varrho_{d_2, d_3} \boxtimes \mu_3)$$

where μ_3 is the Ψ_3 -distribution of $(\hat{\omega}_{3,4} \dots \hat{\omega}_{p,p+1})^2$. We continue this inductive procedure and observe that the last step gives

$$\mu_{p-1} = U_{d_p/d_{p-1}}(\varrho_{d_{p-1}, d_p} \boxtimes \varrho_{d_{p+1}, d_p}),$$

which gives the desired formula for μ_1 since the operation U_s commutes with \boxtimes and we can collect all operations of this type to get $U_{d_2/d_1} \dots U_{d_p/d_{p-1}} = U_{d_p/d_1}$. \blacksquare

Corollary 10.3. *In particular, if $d_1 = d_2 = \dots = d_p = 1$ and $d_{p+1} = t$, the limit distribution of Theorem 10.1 is the free Bessel law $\pi_{p,t}$.*

Proof. This is an immediate consequence of Theorem 10.4. \blacksquare

11. APPENDIX

The definition of the symmetrically matricially free array of units given in [23] should be strenghtened in order that the symmetrized Gaussian operators be symmetrically matricially free. This requires certain changes to be made in [23] which are listed below.

- (1) Condition (2) of Definition 8.1 in [23] should be replaced by condition (2) of Definition 3.4 given in this paper. For that purpose, one needs to distinguish even and odd elements.
- (2) The proof of symmetric matricial freeness of the array of symmetrized Gaussian operators was stated in Proposition 8.1 in [23] and the proof was (unfortunately) omitted. Using Definition 3.4, this fact is proved here in Proposition 3.5.
- (3) Definition 2.3 in [23] should be replaced by Definition 4.3 given in this paper. In principle, it is possible to modify the old definition and use conditions on intersections of unordered pairs, but it is much more convenient and simpler to phrase the new definition using sequences of ordered pairs.

- (4) The proof of Theorem 9.1 in [23] needs to be slightly strengthened since the classes $\mathcal{NC}_{m,q}^2(\{p_1, q_1\}, \dots, \{p_m, q_m\})$ can be smaller, in general, than those considered in [23] since Definition 4.3 is stronger than Definition 2.3 in [23]. Therefore, we need to justify that if $\pi(\gamma) \in \mathcal{NC}_{m,q}^2 \setminus \mathcal{NC}_{m,q}^2(\{p_1, q_1\}, \dots, \{p_m, q_m\})$, then the corresponding mixed moment of symmetric blocks under $\tau_q(n)$ vanishes. Clearly, in order that this moment be non-zero, the imaginary block of $\pi(\gamma)$ must be colored by q . Then all blocks of $\pi(\gamma)$ of depth zero must be labelled by pairs $\{p_i, q_i\}$ which contain q in order that the corresponding symmetric blocks $T_{p_i, q_i}(n)$ act non-trivially on vectors $e_j, j \in N_q$. Supposing that $q_i = q$ for all such pairs $\{p_i, q_i\}$ assigned to the right legs of $\pi(\gamma)$, we set $v_i = (p_i, q_i)$. Then, in turn, all symmetric blocks which correspond to blocks of $\pi(\gamma)$ of depth one whose nearest outer block is labelled by given $\{p_i, q_i\}$ must be labelled by pairs $\{p_k, q_k\}$ which contain p_i . Supposing that $q_k = p_i$ for all such pairs $\{p_k, q_k\}$ assigned to the right legs of $\pi(\gamma)$, we set $v_k = (p_k, q_k)$. We continue in this fashion until all symmetric blocks and the corresponding blocks of $\pi(\gamma)$ are taken into account. Finally, if $\{i, j\}$ is a block, where $i < j$ and $v_j = (p_j, q_j)$, then we must have $v_i = (q_j, p_j)$ in order that the action of $T_{p_i, q_i}(n)$ be non-trivial, which follows from an inductive argument starting from the deepest blocks. Consequently, in order to get a non-zero contribution from the corresponding mixed moment of symmetric blocks, which is proportional to the product of variances as in the proof in [23], there must exist a tuple (v_1, \dots, v_m) such that $v_i \in \{(p_i, q_i), (q_i, p_i)\}$ to which $\pi(\gamma)$ is adapted. Therefore, the conditions of Definition 4.3 are satisfied.

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